On the Complexity of Nesting Polytopes

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Abstract

Given two rational convex polytopes $A \subseteq B \subset \mathbb{R}^d$ and a number k where A is given by vertices and B is given by halfspaces, the Nested Polytope Problem asks whether there exists a polytope X with k vertices such that $A \subseteq X \subseteq B$. We prove that Nested Polytope Problem is $\exists\mathbb{R}$-complete, which implies that Nested Polytope Problem is not contained in the complexity class NP, unless $\exists\mathbb{R} = \text{NP}$. Although this result was, to the best of our knowledge, never pointed out explicitly, it follows from some known results easily, as we will explain [17, 8].

1 Introduction

Given two rational convex polytopes $A \subseteq B \subset \mathbb{R}^d$ and a number k where A is given by vertices and B is given by halfspaces, the Nested Polytope Problem asks whether there exists a polytope X with k vertices such that $A \subseteq X \subseteq B$. The earliest mention of this problem that we know of is by Silio in 1979 [18], who found an $O(nm)$ time algorithm for nesting a triangle between an n-gon and m-gon. Independently Victor Klee suggested the same problem as was pointed out in several papers [11, 2, 9, 16, 10]. Gillis and Glineur showed that the Nested Polytope Problem is polynomial time equivalent to a variant of the Non-negative Matrix Factorization (NMF) problem called Restricted NMF [13]. These problems respectively generalize the Intermediate Simplex problem, where the polytopes A and B are required to be full dimensional and $k = d + 1$, and a special case of NMF called Exact NMF. Vavasis showed that these two problems are polynomial time equivalent to each other, and are NP-hard [19]. Yannakakis showed that NMF is a generalization of the extension complexity problem for polytopes. More specifically, the non-negative rank of the slack-matrix of a polytope corresponds precisely to the extension complexity of the polytope defined by the set of defining linear constraints. Thereby he gave lower bounds on the size of *symmetric* linear programs needed to describe certain combinatorial problems such as the Traveling Salesman problem [20], see also [12] for the asymmetric case. Yannakakis's paper may be celebrated for showing that a swath of fruitless attempts to prove P= NP are untenable. This situation is laid out in Figure 1. Our main contribution is an independent proof that the Nested Polytope Problem is $\exists\mathbb{R}$-complete by a simple geometric construction.

Note that although this seems never to be pointed out explicitly, the result is **not novel**. In 2016, it was shown elegantly by Shitov that NMF is $\exists\mathbb{R}$-complete [17]. Cohen and Rothblum described already in 1993 a simple polynomial reduction from NMF to the Nested Polytope Problem [8].

▶ **Theorem 1.1.** *The* Nested Polytope Problem *is $\exists\mathbb{R}$-complete.*[1]

[1] Our method of proof also implies a universality theorem similar to Mnëv's theorem for oriented matroids, but we do not include it in this abstract.

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This is an extended abstract of a presentation given at EuroCG'19. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

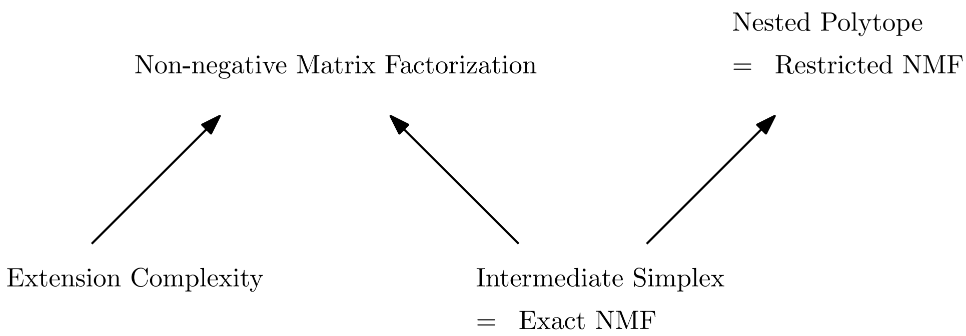


Figure 1 The long and winding road from extension complexity to nested polytopes.

For more work on Nested Polytope Problem and NMF, we refer the reader to the following literature [2, 5, 7, 14, 4, 13, 15, 3, 6].

The proof works in two steps. As a first step, we introduce a variant of the existential theory of the reals, denoted ETR-INV-array, and we show this variant is $\exists\mathbb{R}$-complete. This is described in Section 2. It ensures that we only have to encode algebraic relations that have a specific form. In the second step, we define gadgets, which are small Nested Polytope Problem instances where the coordinates of certain vertices of the nested polytope X are forced to satisfy the algebraic relations from the first step, and then we assemble these small gadgets to define a Nested Polytope Problem instance corresponding to each ETR-INV-array instance.

2 Encoding ETR

As a first step to encode an instance of the existential theory of the reals as an instance of the Nested Polytope Problem, we first simplify the algebraic structure.

An instance $\mathcal{A}$ of ETR-INV-array of size $m \times n$ is an m-by-n matrix A of variables $\alpha_{i,j}$ together with a system of equations of the form

$$\alpha_{i,j} + \alpha_{i,k} = \tfrac{5}{2} \quad , \quad \alpha_{i,j} + \alpha_{i,k} + \alpha_{i,l} = \tfrac{5}{2} \quad , \quad \alpha_{i,k} \cdot \alpha_{j,k} = 1.$$

Note that the linear equations relate variables in the same row and the quadratic equations relate variables in the same column. A solution to $\mathcal{A}$ is an assignment of values $\alpha_{i,j} \in [\frac{1}{2}, 2]^{m \times n}$ to each variable that satisfies each equation of $\mathcal{A}$. The corresponding decision problem asks whether an instance of $\mathcal{A}$ has a solution.

▶ **Lemma 2.1.** ETR-INV-array *is* $\exists\mathbb{R}$*-complete.*

This can be proven by introducing intermediate variables. For example, the relation $\alpha_{i,j} + \alpha_{i,k} = \alpha_{i,l}$ could be obtained by introducing a variable $\alpha_{i,m}$ and using the equations $\alpha_{i,j} + \alpha_{i,k} + \alpha_{i,m} = \frac{5}{2}$ and $\alpha_{i,m} + \alpha_{i,l} = \frac{5}{2}$. A similar reduction is given in [1, Lemma 12].

3 Building the polytopes

This section is devoted to show the following lemma. Together, Lemma 2.1 and Lemma 3.1 establish Theorem 1.1.

▶ **Lemma 3.1.** *Let* $\mathcal{A}$ *be an* ETR-INV*-array of size* $m \times n$*. There exists convex polytopes* $A \subset B \subset \mathbb{R}^{2+n+m}$ *such that there exists a nested polytope* $A \subset X \subset B$ *with* $k = mn + 2m + 2$ *vertices, if and only if* $\mathcal{A}$ *has a solution.*

▶ Remark. In fact we will show something stronger. In our construction, the polytopes $A \subset B$ in Theorem 3.1 will have precisely $2m+2$ vertices in common. It follows that any nested polytope $A \subset X \subset B$ must contain these common vertices. Thus X will have $m \cdot n$ remaining vertices, and our construction will force these remaining vertices to be contained in certain segments of some of the edges of the outer polytope B. By parametrizing each of these segments by the interval $[\frac{1}{2}, 2]$ we obtain a correspondence between the remaining $m \cdot n$ vertices and a subset of $[\frac{1}{2}, 2]^{m \cdot n}$. The key step in the proof of Lemma 3.1 is to show that a positioning of the remaining vertices of X gives us $A \subset X \subset B$, if and only if those vertex positions correspond to a solution of $\mathcal{A}$.

3.1 Two geometric observations

Here we state two simple geometric observations that are used for the "gadgets" needed in our construction of the polytopes of Lemma 3.1. The proofs are simple calculations and left to the reader.

Let $\{v_0, v_1, \ldots, v_k\}$ be a set of linearly independent points in $\mathbb{R}^d$. For $1 \leq i \leq k$ let $w_i = v_i + v_0$ and define the prism P as

$$P = \mathrm{conv}(\{v_1, \ldots, v_k, w_1, \ldots, w_k\}).$$

For $t \in [0,1]$ define the point $q_t \in P$ as

$$q_t = (1-t)(\tfrac{1}{k}v_1 + \cdots + \tfrac{1}{k}v_k) + t(\tfrac{1}{k}w_1 + \cdots + \tfrac{1}{k}w_k) = \tfrac{1}{k}v_1 + \cdots + \tfrac{1}{k}v_k + tv_0.$$

Finally, for $1 \leq i \leq k$ define points p_i as

$$p_i = (1-\lambda_i)v_i + \lambda_i w_i = v_i + \lambda_i v_0,$$

where $\lambda_i \in [0,1]$. We have the following.

▶ **Observation 3.2.** *$q_t \in conv(\{p_1, \ldots, p_k\})$ if and only if $\sum_{i=1}^{k} \lambda_i = tk$.*

▶ **Observation 3.3.** *Let $\alpha_1, \alpha_2 \in [\frac{1}{2}, 2]$ and $p_1 = (\alpha_1, -1)$ and $p_2 = (-1, \alpha_2)$. Then it holds that the origin $(0,0) \in conv(\{p_1, p_2\})$ if and only if $\alpha_1 \cdot \alpha_2 = 1$.*

3.2 A basic outline of the construction

We now give an outline of the construction of the polytopes in Lemma 3.1, without giving explicit coordinates, and rather focusing on the three "gadgets" that will be used to encode the three types of equations in $\mathcal{A}$.

3.2.1 The outer polytope

The outer polytope B is a product of an orthogonal simplex of dimension m with a regular simplex of dimension $n+1$. That is, we start with an *"orthogonal frame"* spanning $\mathbb{R}^m$, consisting of m mutually orthogonal segments of length 3 all meeting in a common point. Note that the convex hull of these segments form an m-dimensional orthogonal simplex. We now take $n+2$ distinct copies of the orthogonal frame, U_1, U_2, $V_1, \ldots, V_n$, each one translated into "independent dimensions" so that their union now lives in $\mathbb{R}^{2+n+m}$. We label the segments of these orthogonal frames as

$$\begin{array}{rcl} U_1 & = & \{\tau_{1,1}, \ldots, \tau_{m,1}\} \\ U_2 & = & \{\tau_{1,2}, \ldots, \tau_{m,2}\} \\ V_1 & = & \{\sigma_{1,1}, \ldots, \sigma_{m,1}\} \\ & \vdots & \\ V_n & = & \{\sigma_{1,n}, \ldots, \sigma_{m,n}\} \end{array}$$

such that the segments $\tau_{i,1}, \tau_{i,2}, \sigma_{i,1}, \ldots, \sigma_{i,n}$ are all parallel.

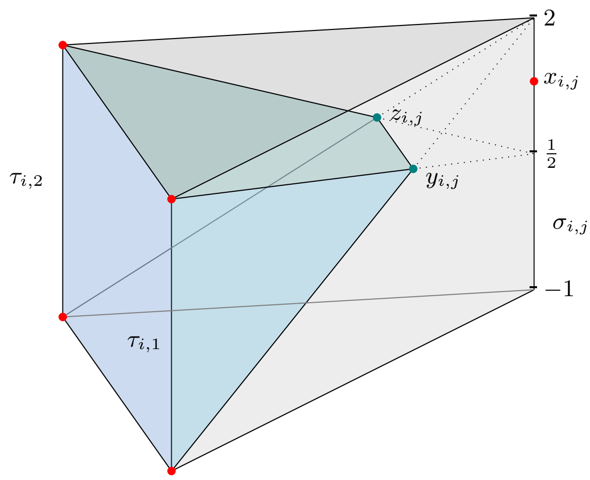
We now take the outer polytope B to be the convex hull of $U_1 \cup U_2 \cup V_1 \cup \cdots \cup V_n$. It is straight-forward to show that B is an $n + m + 1$-dimensional polytope with $(n + 2)(m + 1)$ vertices. In what follows, for each $1 \leq i \leq m$ and $1 \leq j \leq n$, the "second half" of segment $\sigma_{i,j}$, parametrizing the interval $[\frac{1}{2}, 2]$, will correspond to the variable $\alpha_{i,j}$ in ETR-INV-array $\mathcal{A}$. The segments $\tau_{i,j}$ will play an auxiliary role which we now describe.

3.2.2 Building the inner polytope: Enforcing vertices to segments

The first step in building the inner polytope A is to enforce the following.

▶ Property 3.4. Let X be a nested polytope, with $k = mn + 2m + 2$ vertices and $A \subset X \subset B$. For every $1 \leq i \leq m$ and $1 \leq j \leq n$, the segment $\sigma_{i,j} \in V_j$ contains exactly one vertex of X, which we denote by $x_{i,j}$.

(More specifically, each segment of the orthogonal frame V_i will contain exactly one vertex from X in its "second half", thus encoding a value in the interval $[\frac{1}{2}, 2]$.) This can be done as follows. Fix indices $1 \leq i \leq m$ and $1 \leq j \leq n$, and consider segment $\tau_{i,1} \in U_1$ and its parallel copy $\sigma_{i,j} \in V_j$, which are edges of a 2-dimensional face of the outer polytope B. Define the point $y_{i,j}$ to be the unique point in this 2-face such that segment $\tau_{i,1} \in U_1$ is mapped to the second half of its parallel copy $\sigma_{i,j} \in V_j$ by central projection through $y_{i,j}$. Similarly, we define the analogous point $z_{i,j}$ in the 2-face of A spanned by the segment $\tau_{i,2} \in U_2$ and its parallel copy $\sigma_{i,j} \in V_j$. (See Figure 2.)



■ **Figure 2** The vertices of any nested polytope $A \subset X \subset B$ (marked in red) must include the endpoints of segments $\tau_{i,1} \in U_1$ and $\tau_{i,2} \in U_2$, while the last vertex, $x_{i,j}$, must be contained in the segment $\sigma_{i,j} \in V_j$ restricted to the interval $[\frac{1}{2}, 2]$.

At this stage of the construction the inner polytope A will consist of the orthogonal frames U_1 and U_2 together with the points $\{y_{i,j}, z_{i,j}\}$ for all $1 \le i \le m$ and $1 \le j \le n$. Moreover, if X is a nested polytope, with $mn + 2m + 2$ vertices and $A \subset X \subset B$, then X must contain the orthogonal frames U_1 and U_2 (which accounts for $2m + 2$ of the vertices) and one vertex in each of the segments of the orthogonal frames $V_1, \dots, V_n$ (accounting for the remaining $m \cdot n$ vertices). Thus Property 3.4 is satisfied, and we let $x_{i,j}$ denote the unique vertex of X which is contained in the (second half of the) segment $\sigma_{i,j} \in V_j$, which we associate with the variable $\alpha_{i,j} \in [\frac{1}{2}, 2]$.

3.2.3 Building the inner polytope: Encoding the linear equations

In order to enforce the relation $\alpha_{i,j} + \alpha_{i,k} = \frac{5}{2}$, we add a new vertex $p_{i,j,k}$ to the inner polytope A as follows. We consider the rectangular 2-face of the outer polytope B spanned by the segments $\sigma_{i,j} \in V_j$ and $\sigma_{i,k} \in V_k$. Define $p_{i,j,k}$ to be the point in this 2-face such that $p_{i,j,k}$ is contained in the convex hull of the vertices $x_{i,j}$ and $x_{i,k}$ of the nested polytope X (satisfying Property 3.4) if and only if the associated variables $\alpha_{i,j} + \alpha_{i,k} = \frac{5}{2}$. (The unique point $p_{i,j,k}$ exists by Observation 3.2. See Figure 3.)

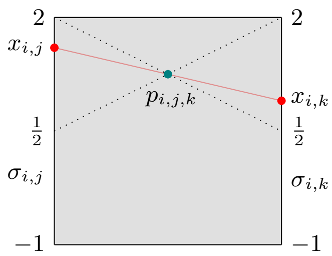


Figure 3 The vertices $x_{i,j}$ and $x_{i,k}$ contain the point $p_{i,j,k}$ in their convex hull if and only if the associated variables satisfy the equation $\alpha_{i,j} + \alpha_{i,k} = \frac{5}{2}$

By adding the vertex $p_{i,j,k}$ to A, it follows that for any nested polytope X satisfying Property 3.4, the associated variables satisfy the equation $\alpha_{i,j} + \alpha_{i,k} = \frac{5}{2}$.

Enforcing the relation $\alpha_{i,j} + \alpha_{i,k} + \alpha_{i,l} = \frac{5}{2}$ is similar to the previous case, and we add a new vertex $q_{i,j,k,l}$ to the inner polytope A as follows. We consider the trianglular prism spanned by the segments $\sigma_{i,j} \in V_j$, $\sigma_{i,k} \in V_k$, and $\sigma_{i,l} \in V_l$, which is a 3-face of the outer polytope B.

Define $q_{i,j,k,l}$ to be the point in this 3-face such that $q_{i,j,k,l}$ is contained in the convex hull of the vertices $x_{i,j}$, $x_{i,k}$, and $x_{i,l}$ of the nested polytope X (satisfying Property 3.4) if and only if the associated variables $\alpha_{i,j} + \alpha_{i,k} + \alpha_{i,l} = \frac{5}{2}$. (The unique point $q_{i,j,k,l}$ exists by Observation 3.2. See Figure 4.)

By adding the vertex $q_{i,j,k,l}$ to A, it follows that for any nested polytope X satisfying Property 3.4, the associated variables satisfy the equation $\alpha_{i,j} + \alpha_{i,k} + \alpha_{i,l} = \frac{5}{2}$.

3.2.4 Building the inner polytope: Encoding the quadratic equations

In order to enforce the relation $\alpha_{i,k} \cdot \alpha_{j,k} = 1$ we add a new vertex $r_{i,j,k}$ to the inner polytope A as follows. Consider the triangular 2-face of B spanned by segments $\sigma_{i,k} \in V_k$ and $\sigma_{j,k} \in V_k$. We can coordinatize the plane containing this 2-face such that the segment $\sigma_{i,k}$ is parametrized by $\{(x, -1) : -1 \le x \le 2\}$ and the segment $\sigma_{j,k}$ is parametrized by $\{(-1, y) : 1 \le y \le 2\}$. We then define $r_{i,j,k}$ to be the origin with respect to this coordinate

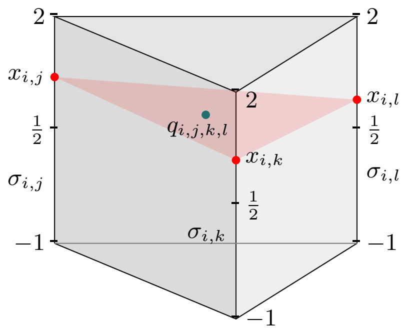


Figure 4 The vertices $x_{i,j}$, $x_{i,k}$, and $x_{i,l}$ contain the point $q_{i,j,k,l}$ if and only if the associated variables satisfy the equation $\alpha_{i,j} + \alpha_{i,k} + \alpha_{i,l} = \frac{5}{2}$.

system. It follows from Observation 3.3 that the vertices $x_{i,k}$ and $x_{j,k}$ contain the point $r_{i,j,k}$ in their convex hull if and only if the associated coordinates satisfy the equation $\alpha_{i,k} \cdot \alpha_{j,k} = 1$. (See Figure 5.)

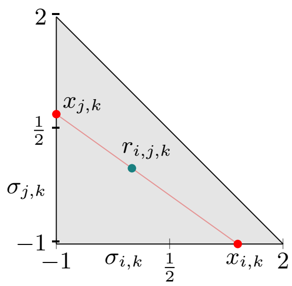


Figure 5 The vertices $x_{i,k}$ and $x_{j,k}$ contain the point $r_{i,j,k}$ if and only if the associated variables satisfy the equation $\alpha_{i,k} \cdot \alpha_{j,k} = 1$.

By adding the vertex $r_{i,j,k}$ to A, it follows that for any nested polytope X satisfying Property 3.4, the associated variables satisfy the equation $\alpha_{i,k} \cdot \alpha_{j,k} = 1$.

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