Extending Simple Drawings*

Alan Arroyo[1], Martin Derka[2], and Irene Parada[3]

1 **IST Austria, Klosterneuburg, Austria**
`alanmarcelo.arroyoguevara@ist.ac.at`
2 **Carleton University, Ontario, Canada**
`mderka@uwaterloo.ca`
3 **Graz University of Technology, Graz, Austria**
`iparada@ist.tugraz.at`

Abstract

Simple drawings are those in which (i) every pair of edges have at most one point in common, and it is either an endpoint or a proper crossing; and (ii) no three edges cross in the same point. In this paper we study the problem of extending a simple drawing $D(G)$ of a graph $G = (V, E)$, by adding a set of edges (of the complete graph with vertex set V) such that the result is a simple drawing with $D(G)$ as a subdrawing. In the context of rectilinear drawings, the problem is trivial. In contrast, we prove that finding the maximum amount of edges from a prescribed set that extend a simple drawing is NP-hard.

1 Introduction

A *simple drawing* of a graph G (also known as *good drawing* or as *simple topological graph* in the literature) is a drawing $D(G)$ of G in the plane such that every pair of edges share at most one point that is either a proper crossing (no tangent edges allowed) or an endpoint. Moreover, no three edges intersect in the same point and edges must not contain other vertices. In some contexts, such as the study of crossing numbers, simple drawings play a central role. Despite them being widely studied, there are basic aspects that remain unknown.

The long-standing conjectures on the crossing numbers of K_n and $K_{n,m}$, known as the Harary-Hill and Zarankiewicz's conjectures, respectively, have drawn particular interest in the study of simple drawings of complete and complete bipartite graphs. Although these problems remain open, their intensive study has produced deep results about simple drawings of K_n [6, 9] and $K_{n,m}$ [2].

In contrast to what we know about K_n, little is known about simple drawings of general graphs. In [8] it was observed that, when studying simple drawings of general graphs, it would be natural to try extend them, by adding the missing edges between non-adjacent vertices, to simple drawings of complete graphs. One of the main results in this paper suggests that there is no hope on efficiently deciding when such closure operation can be performed.

Given a simple drawing $D(G)$ of a graph $G = (V, E)$, and a set M of edges of the complete graph with vertex set V, an *extension* of $D(G)$ with a set of edges M is a simple drawing $D'(H)$ of the graph $H = (V, E \cup M)$ that contains $D(G)$ as a subdrawing. If that extension exists we say the the edge uv can be *added* to $D(G)$. An extension with one given edge is not always possible, as shown by Kynčl [7] (in Figure 1a the edge uv cannot be added). We can extend this example to a simple drawing of $K_{2,4}$ (Figure 1b) and we can use this to construct larger drawings of $K_{n,m}$ in which an edge uv cannot be added. Moreover, Kyncl's drawing can be extended to a simple drawing of K_6 missing an edge that cannot be added

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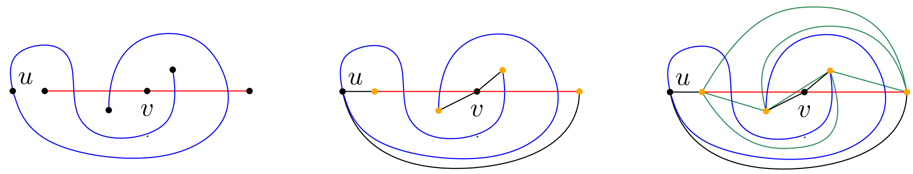


(a) Example by Kynčl [7]. **(b)** Drawing of $K_{2,4}$. **(c)** Drawing of $K_6 \setminus uv$.

Figure 1 Drawings that cannot be extended with the edge uv.

(Figure 1c), and again we can use this drawing to construct larger drawings of K_n missing an edge that cannot be added.

Extensions have been previously considered in the context of *saturated* simple drawings, that is, drawings where no edge can be added [8, 4]. In the context of saturated drawings, the main interest is on finding the minimum number of edges that a saturated graph on n vertices can have. This minimum was first shown to be at most $17.5n$ [8] and later $7n$ [4].

In this paper, we focus on extensions of simple drawings of general graphs. In Section 2 we show that given a simple drawing $D(G)$ of a graph $G = (V, E)$ and a set M of edges of the complete graph with vertex set V and with $M \cap E = \emptyset$, it is NP-hard to find the maximum subset of edges from M that can be added to $D(G)$. In the full version of the paper we also study the case in which only one edge is to be added. In Section 3 we discuss these results and present open questions.

2 Hardness of Extending Simple Drawings

In this section we prove the following result:

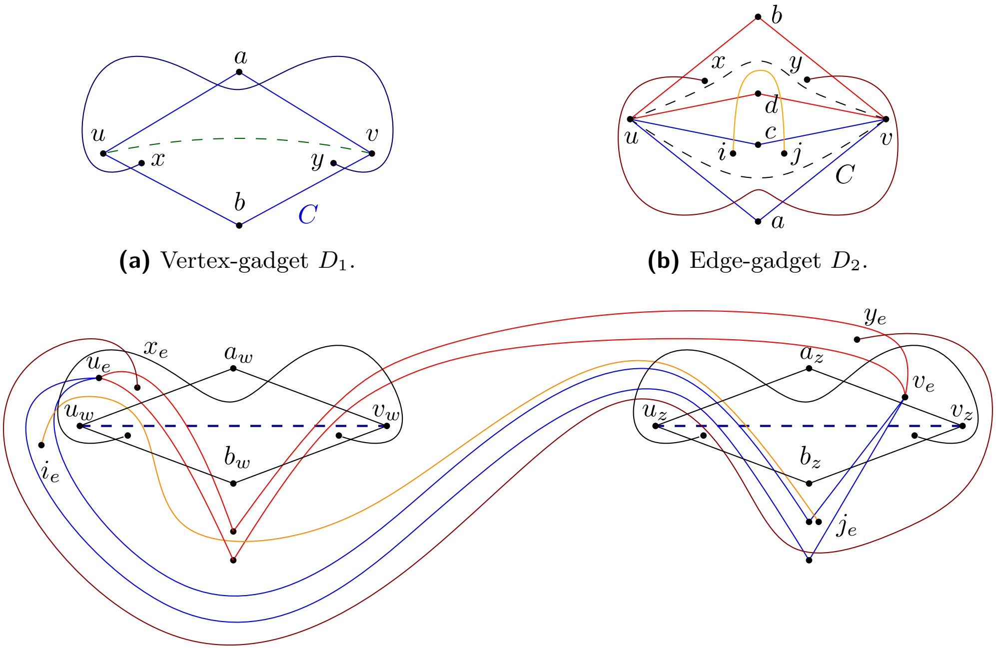
▶ **Theorem 2.1.** *Given a simple drawing $D(G)$ of a graph $G = (V, E)$ and a set M of edges of the complete graph with the vertex set V and with $E \cap M = \emptyset$, it is NP-hard to find a maximum subset of edges $M' \subseteq M$ that extends $D(G)$.*

Our proof of Theorem 2.1 is based on a reduction from the maximum independent set problem (MIS). An independent set of a graph $G = (V, E)$ is a set of vertices $S \subseteq V$ such that no two vertices in S are incident to the same edge. The problem of determining the maximum independent set (MIS) of a given graph is NP-hard in general, and it remains NP-hard when the input is a planar graph with maximum degree 3 [3, Lemma 1]. We first describe the construction of a simple drawing $D'(G')$ given an MIS instance. Then we argue that for a well selected set of edges M that are not present in $D'(G')$, finding a maximum subset $M' \subseteq M$ that can simultaneously extend $D'(G')$ is equivalent to finding a maximum independent set in the input instance.

2.1 Constructing a drawing from a given graph

We begin by introducing our two basic gadgets D_1 and D_2 (shown in Figure 2). The vertex gadget D_1 consists of a cycle C on four vertices a, v, b, u drawn in the plane without any crossings. We add two additional vertices x and y to its interior and connect them with an edge that, starting in x crosses edge bu to the exterior of C, continues through ua to the interior of C, crosses av to the exterior of C, and vb to the interior of C where it ends in y.

The drawing D_1 has the property that the only way of adding edge uv is by following an arc such as the dashed one depicted in Figure 2a (with maybe also crossing the edge xy, but



(a) Vertex-gadget D_1.

(b) Edge-gadget D_2.

(c) Constructed drawing obtained by a reduction from the path graph on two vertices w and z.

Figure 2 Basic gadgets and drawings.

staying in the same region). Routing through the exterior of C would force either a double crossing with edge xy, or a crossing with an edge incident to u or v.

The edge gadget D_2 is obtained by adding additional vertices to the interior of D_1. Specifically, we add two vertices d and c and edges ud, vd and uc, vc that are drawn so that $udvc$ is a crossing-free cycle in the interior of C. Note that since it is crossing-free, the vertices x and y are in the interior of cycle $ubvd$. We add two more vertices, called i and j, in the interior of $ucva$ and we connect them with an edge that after starting in i, crosses edges uc, ud, vd, vc, and ends in j (without crossing any other edges). See also Figure 2b.

Similarly as in the case of D_1, to extend D_2 into a simple drawing with the edge uv, the edge needs to be routed through the interior of C either in the interior of cycle $ubvd$ or of $ucva$ as depicted in Figure 2b (with maybe also crossing the edge xy or the edge ij but staying in the same region). Furthermore, it cannot be routed through the interior of $udvc$ as it would need to intersect either an edge incident to u or v, or cross the edge ij twice.

In Figure 2c we can see a combination of an edge gadget and two vertex gadgets: it shows a copy D_2^e of the gadget D_2 (that we will say *corresponds* to an edge $e := wz$) over two different copies, D_1^w and D_1^z, of the gadget D_1 (that we will say *correspond* to vertices w and z, respectively). Notice that we add the label of the vertex or edge corresponding to the gadget (in this case either w or z or e) as a superindex. Since the region where both $v_w u_w$ and $v_z u_z$ can be drawn is forced, adding both prevents $v_e u_e$ from being added. Adding either only edge $v_w u_w$ or only edge $v_z u_z$ leaves exactly one possible region for edge $v_e u_e$.

We have all the main ingredients for our construction. Suppose that we are given a planar graph $G = (V, E)$ with maximum degree at most 3. This graph admits a 2-page book embedding $D(G)$ [5, 1]. In a 2-page book embedding all the vertices are placed on a (horizontal) line and the edges are arcs lying either in the upper half-plane on in the lower one and there are no proper crossings. The following lemma shows that replacing each vertex

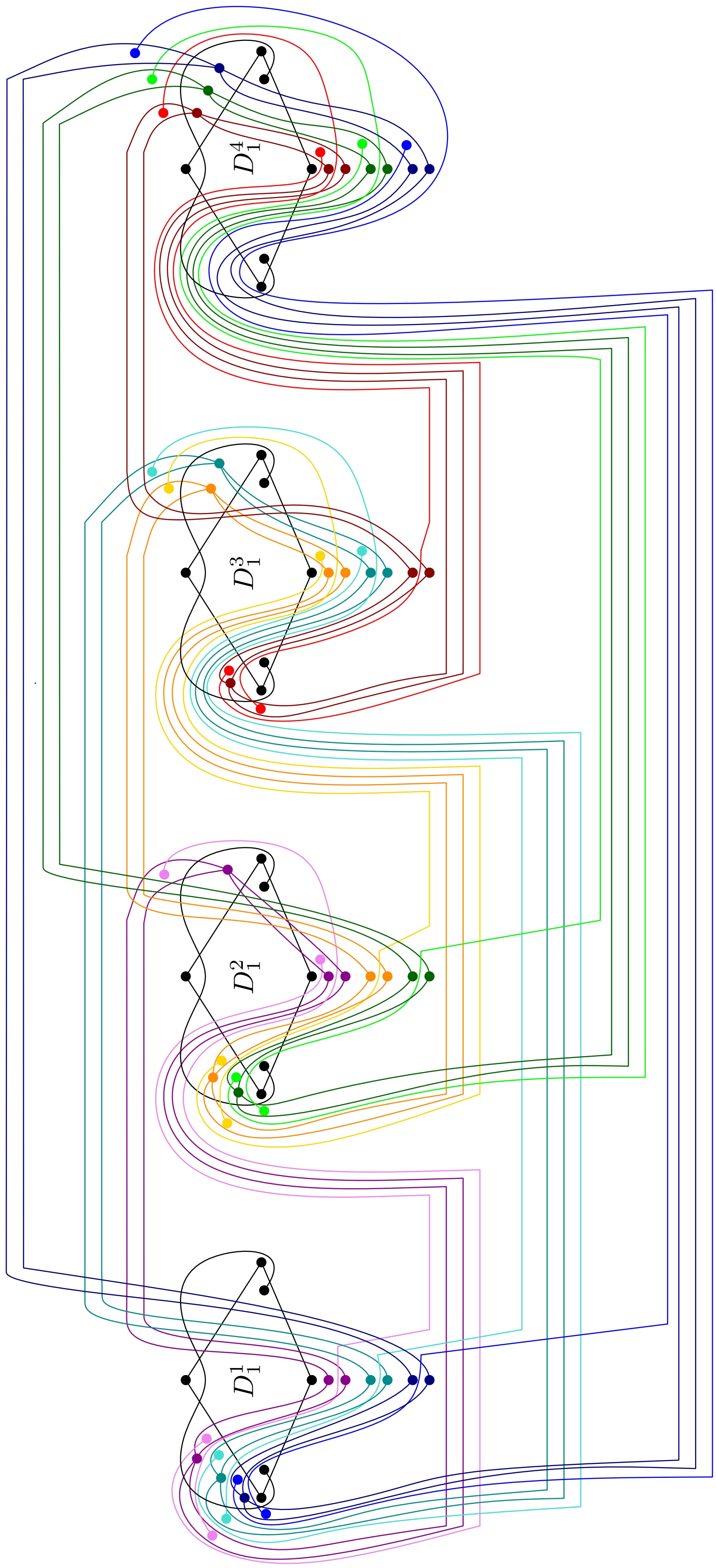


Figure 3 Drawing obtained by a reduction from K_4.

$w \in V$ in the drawing by a vertex gadget D_1^w and each edge $e \in E$ by an edge gadget D_2^e, we construct a simple drawing $D'(G')$.

▶ **Lemma 2.2.** *Given a 2-page book embedding $D(G)$ of a graph $G = (V, E)$, we can replace every vertex by a vertex gadget and every edge by an edge gadget to obtain a simple drawing.*

Proof. We will show that the copies $\{D_2^e : e \in E\}$ can be added to $\bigcup_{w \in V} D_1^w$ such that for every edge $e \in E$ incident to w and z ($w, z \in V$), $D_1^w \cup D_1^z \cup D_2^e$ is as in Figure 2c (up to interchanging the indices w and z), and the resulting drawing is a simple drawing.

First, for each vertex $w \in V$ we place the gadget D_1^w in its position, so all the copies of D_1 lie (equidistant) in a horizontal line. For the edges of G, since the drawing in Figure 2c is not symmetric, we choose an orientation. We orient all the edges in the 2-page book embedding $D(G)$ from left to right. We start adding the corresponding D_2 gadgets from left to right and from the shortest edges to the longest (where the length is the Euclidean distance between the endpoints). For an edge wz the intersections of the gadget D_2^{wz} (i) with the edges $u_w a_w$ and $u_w b_w$ are placed to the left of all the previous intersections of other edge gadgets with that edge; (ii) with the edge $v_w b_w$ are placed to the right of all the previous intersections with that edge; (iii) with the edge $v_w a_w$ are placed to the right of previous intersections with gadgets D_2^{wt} and to the left of previous intersections with gadgets D_2^{tw}; (iv) with the edges $u_z a_z$ and $u_z b_z$ are placed to the left of the previous intersections with gadgets D_2^{tz} (v) with the edge $v_z b_z$ are placed to the left of all previous intersections; and (vi) with the edge $v_z a_z$ are placed to the left of all previous intersections with gadgets D_2^{tz}. See Figure 3.

Moreover, the segments of some of the edges in the edge gadgets connecting from one vertex gadget to another vertex gadget can be drawn as strips in either the upper or lower half-plane with respect to the horizontal line. In those strips, segments of edges in the same strip don't cross and segments of edges in different strips cross at most once. See Figure 3.

Since neither of the gadgets of two incident edges cross, and edges between different gadgets are vertex-disjoint, we only have to worry about edges from different gadgets crossing more than once. By construction, no edge in an edge gadget intersects more than once with an edge in a vertex gadget. Thus, it remains to show that any two edges e_1 and e_2 from two distinct gadgets cross at most once. Such two edges are included in a subgraph H of G with exactly four vertices. The drawing induced by the four vertex gadgets and the at most six edge gadgets is homeomorphic to a subdrawing of the drawing in Figure 3. It is routine to check that this drawing a simple drawing, and thus e_1 and e_2 cross at most once. ◀

2.2 Reduction from Maximum Independent Set

For the decision version of the problem, given a planar graph $G = (V, E)$ with vertex degree at most 3 and a constant k, we reduce the problem of deciding if G has an independent set of size k to the problem of deciding if the simple drawing $D'(G')$ with a candidate set of edges M (where $M = \{u_w v_w : w \in V\} \cup \{u_e v_e : e \in E\}$) can be extended with a set of edges $M' \subseteq M$ with cardinality $|M'| = |E| + k$.

▶ **Lemma 2.3.** *The construction exhibited in the previous subsection is a polynomial-time reduction from independent set in planar maximum degree 3 graphs.*

Proof. To show the correctness of the (polynomial) reduction we first show that if G has an independent set I of size k then we can extend $D'(G')$ with a set M' of $|E| + k$ edges of M. Clearly, the k edges $\{u_w v_w : w \in I\}$ can be added to $D'(G')$ by the construction of the gadgets. Since I is an independent set, each edge has at most one endpoint in I. Thus, in every edge gadget D_2^e at most one of the two possibilities for adding the edge $u_e v_e$ is blocked by the previous k added edges. We therefore can also add the $|E|$ edges $\{u_e v_e : e \in E\}$.

Conversely, assume that the set $M' \subset M$ of $|E| + k$ edges can be added to $D'(G')$. If the set of vertices $\{w : u_w v_w \in M'\}$ is an independent set of G, then we are done, since at most $|E|$ edges of the added ones can be from edge gadgets, so at least k are from vertex gadgets. Otherwise, there are two edges $u_w v_w$ and $u_z v_z$ in M' such that the corresponding vertices $w, z \in V$ are connected by the edge $wz \in E$. This implies that the edge $u_{wz} v_{wz}$ belongs to M but it cannot be in M'. By removing the edge $u_w v_w$ and adding the edge $u_{wz} v_{wz}$ to $D'(G')$ we obtain another valid extension with the same cardinality but one less edge belonging to a vertex gadget. Iteratively repeating this, we end with an extension N of $D'(G')$ that has cardinality $|E| + k$ and such that the set of vertices $\{w : u_w v_w \in N\}$ is an independent set of G of size at least k. ◀

3 Conclusions

In this paper we showed that, given a simple drawing $D(G)$ of a graph $G = (V, E)$ and a prescribed set M of edges of the complete graph with vertex set V, it is NP-hard to find the maximum number of edges from M that can be added to $D(G)$ such that the resulting drawing is simple. Focusing on the case $|M| = 1$, in the full version of this paper, on the one hand, we considered the problem in a dual setting and showed that this slight generalization is NP-complete and, on the other hand, we found sufficient conditions guaranteeing a polynomial-time decision. We hope that the work done in this direction paves the way to show the following:

▶ **Conjecture 1.** *Given a simple drawing $D(G)$ of a graph G and a pair u, v of non-adjacent edges, we can decide in polynomial time whether we can add uv to $D(G)$.*

Finally, modifying both the examples in Figure 1 and a previous example in [8, Figure 11] one can obtain arbitrarily large non-extensible drawings (where a given edge cannot be added) of graphs including complete bipartite graphs, complete graphs missing one edge, and matchings. Moreover, a modification of [8, Figure 1] shows that there are arbitrarily large examples that cannot be extended with an edge but such that the removal of any vertex or edge allows it to be extensible with any missing edge. So there is no hope of characterizing non-extensible drawings in terms of subdrawings. It is also not true that any graph with no isolated points has a non-extensible drawing, as any drawing of $K_{1,m}$ can be extended with any missing edge. This motivates the following problem:

▶ **Problem 1.** *Characterize all graphs that admit a non-extensible drawing.*

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