Reliable Geometric Spanners

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Abstract

We show how to construct a $(1+\varepsilon)$-spanner over a set P of n points in $\mathbb{R}^d$ that is resilient to a catastrophic failure of nodes. Specifically, for prescribed parameters $\vartheta, \varepsilon \in (0,1)$, the computed spanner G has $\mathcal{O}\big(\varepsilon^{-7d} \log^7 \varepsilon^{-1} \cdot \vartheta^{-6} n \log n (\log\log n)^6\big)$ edges. Furthermore, for *any* k, and *any* deleted set $B \subseteq P$ of k points, the residual graph $G \setminus B$ is a $(1+\varepsilon)$-spanner for all the points of P except for $(1+\vartheta)k$ of them. No previous constructions, beyond the trivial clique with $\mathcal{O}(n^2)$ edges, were known such that only a tiny additional fraction (i.e., ϑ) lose their distance preserving connectivity.

1 Introduction

Spanners. A Euclidean graph is a graph whose vertices are points in $\mathbb{R}^d$ and the edges are weighted by the Euclidean distance between their endpoints. Let $G = (P, E)$ be a Euclidean graph and $p, q \in P$ be two vertices of G. For a parameter $t \geq 1$, a path between p and q in G is a ***t-path*** if the length of the path is at most $t\,\|p-q\|$, where $\|p-q\|$ is the Euclidean distance between p and q. The graph G is a ***t-spanner*** of P if there is a t-path between any pair of points $p, q \in P$. We denote the length of the shortest path between $p, q \in P$ in the graph G by $\mathsf{d}(p,q)$.

Spanners have been studied extensively. The main goal in spanner constructions is to have small *size*, that is, to use as few edges as possible. Other desirable properties are low degrees [1, 8, 15], low weight [5, 10], low diameter [2, 3] or to be resistant against failures [6, 11, 12, 13]. The book by Narasimhan and Smid [14] gives a comprehensive overview.

Robustness. In this paper, our goal is to construct spanners that are robust according to the notion introduced by Bose *et al.* [6]. Intuitively, a spanner is robust if the deletion of k vertices only harms a few other vertices. Formally, a graph G is an $f(k)$-robust t-spanner, for some positive monotone function f, if for any set B of k vertices deleted in the graph, the remaining graph $G \setminus B$ is still a t-spanner for at least $n - f(k)$ of the vertices. Note, that the graph $G \setminus B$ has $n-k$ vertices – namely, there are at most $\mathcal{L}(k) = f(k) - k$ additional vertices that no longer have good connectivity to the remaining graph. The quantity $\mathcal{L}(k)$ is the ***loss***. We are interested in minimizing the loss.

The natural question is how many edges are needed to achieve a certain robustness (since the clique has the desired property). That is, for a given parameter t and function f, what is the minimal size that is needed to obtain an $f(k)$-robust t-spanner on any set of n points.

A priori it is not clear that such a sparse graph should exist (for t a constant) for a point set in $\mathbb{R}^d$, since the robustness property looks quite strong. Surprisingly, Bose *et al.* [6] showed that one can construct a $\mathcal{O}(k^2)$-robust $\mathcal{O}(1)$-spanner with $\mathcal{O}(n \log n)$ edges. Bose *et al.* [6] proved various other bounds in the same vein on the size for one-dimensional and higher-dimensional point set. Their most closely related result is that for the one-dimensional

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point set $P = \{1, 2, \ldots, n\}$ and for any $t \geq 1$ at least $\Omega(n \log n)$ edges are needed to construct an $\mathcal{O}(k)$-robust t-spanner.

ϑ-reliable spanners. We are interested in building spanners where the loss is only fractional. Specifically, given a parameter ϑ, we consider the function $f(k) = (1+\vartheta)k$. The loss in this case is $\mathcal{L}(k) = f(k) - k = \vartheta k$. A $(1+\vartheta)k$-robust t-spanner is a ***ϑ-reliable t-spanner***.

Exact reliable spanners. If the input point set is in one dimension, then one can easily construct a 1-spanner for the points, which means that the exact distances between points on the line are preserved by the spanner. This of course can be done easily by connecting the points from left to right. It becomes significantly more challenging to construct such an exact spanner that is reliable.

1.1 Our results

We investigate how to construct reliable spanners with very small loss – that is ϑ-reliable spanners. To the best of our knowledge nothing was known on this case before this work.

(A) **Exact reliable spanners in one dimension.** We show how to construct an $\mathcal{O}(1)$-reliable exact spanner on any one-dimensional set of n points with $\mathcal{O}(n \log n)$ edges. The idea of the construction is to build a binary tree over the points, and to build bipartite expanders between certain subsets of nodes in the same layer. One can think of this construction as building different layers of expanders for different resolutions. The construction is described in Section 3. See Theorem 3.3 for the result.
One can get added redundancy by systematically shifting the layers. Done carefully, this results in a ϑ-reliable exact spanner. See Theorem 3.4 for the result.

(B) **Reliable $(1+\varepsilon)$-spanners in higher dimensions.** We next show a surprisingly simple and elegant construction of ϑ-reliable spanners in two and higher dimensions, using a recent result of Chan *et al.* [9], which show that one needs to maintain only a "few" linear orders. This immediately reduces the d dimensional problem to maintaining a reliable spanner for each of this orderings, which is the problem we already solved. See Section 4 for details.

Omitted proofs and more results can be found in the full version [7]. A very recent result of Bose *et al.* [4] obtains similar bounds on the size of reliable spanners in higher dimensions.

2 Preliminaries

For a set X of vertices in a graph $G = (V, E)$, let $\Gamma(X) = \left\{ v \in V \;\middle|\; uv \in E \text{ for a } u \in X \right\}$ be the ***neighbors*** of X in G. The following lemma, which is a standard expander construction, provides the main building block of our one-dimensional construction.

▶ **Lemma 2.1.** *Let L, R be two disjoint sets, with a total of n elements, and let $\xi \in (0,1)$ be a parameter. One can build a bipartite graph $G = (L \cup R, E)$ with $\mathcal{O}(n/\xi^2)$ edges, such that*

(I) for any subset $X \subseteq L$, with $|X| \geq \xi |L|$, we have that $|\Gamma(X)| > (1-\xi)|R|$, and

(II) for any subset $Y \subseteq R$, with $|Y| \geq \xi |R|$, we have that $|\Gamma(Y)| > (1-\xi)|L|$.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and let $[i : j] = \{i, i+1, \ldots, j\}$. Our purpose is to build a reliable 1-spanner in one dimension. Intuitively, a point in $[n]$ is in trouble, if many of its close by neighbors belong to the failure set B. Such an element is in the shadow of B, defined formally next.

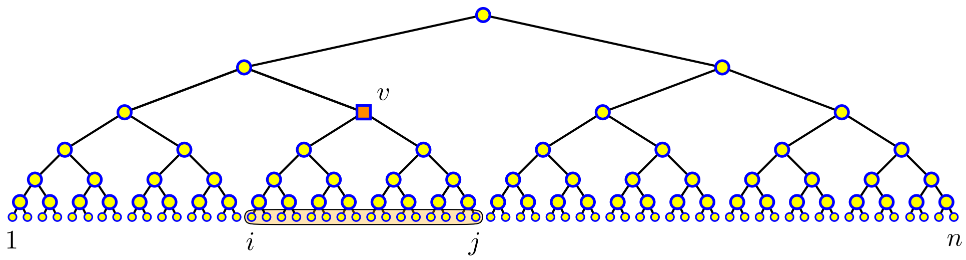

Figure 1 The binary tree built over $[n]$. The block of node v is the interval $[i : j]$.

▶ **Definition 2.2.** Consider an arbitrary set $B \subseteq [n]$ and a parameter $\alpha \in (0,1)$. A number i is in the ***left*** α***-shadow*** of B, if and only if there exists an integer $j \geq i$, such that $\left|[i : j] \cap B\right| \geq \alpha \left|[i : j]\right|$. Similarly, i is in the ***right*** α***-shadow*** of B, if and only if there exists an integer i, such that $h \leq i$ and $|[h : i] \cap B| \geq \alpha \, |[h : i]|$. The left and right α-shadow of B is denoted by $\mathcal{S}_{\rightarrow}(B)$ and $\mathcal{S}_{\leftarrow}(B)$, respectively. The combined shadow is denoted by $\mathcal{S}(\alpha, B) = \mathcal{S}_{\rightarrow}(B) \cup \mathcal{S}_{\leftarrow}(B)$.

▶ **Lemma 2.3.** *Fix a set $B \subseteq [n]$ and let $\alpha \in (0,1)$ be a parameter. Then, we have that $|\mathcal{S}_{\rightarrow}(B)| \leq (1 + \lceil 1/\alpha \rceil) \, |B|$. In particular, the size of $\mathcal{S}(\alpha, B)$ is at most $2(1 + \lceil 1/\alpha \rceil) \, |B|$.*

3 Reliable spanners in one dimension

3.1 Constructing the graph H

Assume n is a power of two, and consider building the natural full binary tree T with the numbers of $[n]$ as the leaves. Every node v of T corresponds to an interval of numbers of the form $[i : j]$ its canonical interval, which we refer to as the block of v, see Figure 1. Let $\mathcal{I}$ be the resulting set of all blocks. In each level one can sort the blocks of the tree from left to right. Two adjacent blocks of the same level are neighbors. For a block $I \in \mathcal{I}$, let $\mathrm{next}(I)$ and $\mathrm{prev}(I)$ be the blocks (in the same level) directly to the right and left of I, respectively. We build the graph of Lemma 2.1 with $\xi = 1/16$ for any two neighboring blocks in $\mathcal{I}$. Let H be the resulting graph when taking the union over all the sets of edges generated by the above.

3.2 Analysis

In the following we show that the resulting graph H is an $\mathcal{O}(1)$-reliable 1-spanner on $\mathcal{O}(n \log n)$ edges. We start by verifying the size of the graph.

▶ **Lemma 3.1.** *The graph H has $\mathcal{O}(n \log n)$ edges.*

Proof. Let $h = \log n$ be the depth of the tree T. In each level $i = 1, 2, \ldots, h$ of T there are 2^{h-i} nodes and the blocks of these nodes have size 2^i. The number of pairs of adjacent blocks in level i is $2^{h-i} - 1$ and each pair contributes $\mathcal{O}(2^i)$ edges. Therefore, each level of T contributes $\mathcal{O}(n)$ edges. We get $\mathcal{O}(n \log n)$ for the overall size by summing up for all levels. ◀

Given two numbers i and j, where $i < j$, consider the two blocks $I, J \in \mathcal{I}$ that correspond to the two numbers at the bottom level. Set $I_0 = I$, and $J_0 = J$. We now describe a canonical walk from I to J, where initially $\ell = 0$. During the walk we have two active blocks I_ℓ and

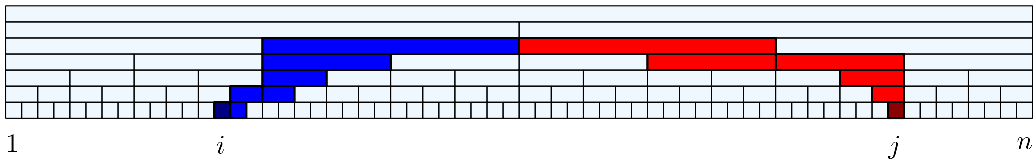

Figure 2 The canonical path between the vertices i and j. The blue blocks correspond to the ascent part and the red blocks correspond to the descent part of the walk.

J_ℓ, that are both in the same level. For any block $I \in \mathcal{I}$ we denote its parent by $p(I)$. At every iteration we bring the two active blocks closer to each other by moving up in the tree.

Specifically, repeatedly do the following:

(A) If I_ℓ and J_ℓ are neighbors then the walk is done.

(B) If I_ℓ is the right child of $p(I_\ell)$, then set $I_{\ell+1} = \mathrm{next}(I_\ell)$ and $J_{\ell+1} = J_\ell$, and continue to the next iteration.

(C) If J_ℓ is the left child of $p(J_\ell)$, then set $I_{\ell+1} = I_\ell$ and $J_{\ell+1} = \mathrm{prev}(J_\ell)$, and continue to the next iteration.

(D) Otherwise – the algorithm ascends. It sets $I_{\ell+1} = p(I_\ell)$, and $I_{\ell+1} = p(J_\ell)$, and it continues to the next iteration.

It is easy to verify that this walk is well defined, and let

$$\pi(i,j) \equiv \underbrace{I_0 \to I_1 \to \cdots \to I_\ell}_{\text{ASCENT}} \to \underbrace{J_\ell \to \cdots \to J_0}_{\text{DESCENT}}$$

be the resulting walk on the blocks where we removed repeated blocks. Figure 2 illustrates the path of blocks between two vertices i and j.

In the following, consider a fixed set $B \subseteq [n]$ of faulty nodes. A block $I \in \mathcal{I}$ is α-***contaminated***, for some $\alpha \in (0,1)$, if $|I \cap B| \geq \alpha\,|I|$.

▶ **Lemma 3.2.** *Consider two nodes $i, j \in [n]$, with $i < j$, and let $\pi(i,j)$ be the canonical path between i and j. If any block of $\pi = \pi(i,j)$ is α-contaminated, then i or j are in the $\alpha/3$-shadow of B.*

▶ **Theorem 3.3.** *The graph H constructed above on the set $[n]$ is an $\mathcal{O}(1)$-reliable exact spanner and has $\mathcal{O}(n \log n)$ edges.*

3.3 ϑ-reliable exact spanners

We can extend Theorem 3.3, to build a one dimensional graph H_ϑ, such that for any fixed $\vartheta > 0$ and any set B of k deleted vertices, at most $(1+\vartheta)k$ vertices are no longer connected by a 1-path after the removal of B. The basic idea is to retrace the construction of Theorem 3.3, and extend it to this more challenging case. The main new ingredient is a shifting scheme.

▶ **Theorem 3.4** ([7]). *For parameters n and $\vartheta > 0$, the graph H_ϑ constructed over $[n]$, is a ϑ-reliable exact spanner. Furthermore, H_ϑ has $\mathcal{O}(\vartheta^{-6} n \log n)$ edges.*

4 Building a reliable spanner in $\mathbb{R}^d$

In the following, we assume that $P \subseteq [0,1)^d$ – this can be done by an appropriate scaling and translation of space. For an ordering σ of $[0,1)^d$, and two points $p, q \in [0,1)^d$, such that $p \prec q$, let $(p,q)_\sigma = \left\{ z \in [0,1)^d \;\middle|\; p \prec z \prec q \right\}$ be the set of points between p and q in the order σ. We need the following minor variant of a result of Chan *et al.* [9].

▶ **Theorem 4.1** ([9]). *For $\varsigma \in (0,1)$, there is a set $\Pi^+(\varsigma)$ of $M(\varsigma) = \mathcal{O}(\varsigma^{-d} \log \varsigma^{-1})$ orderings of $[0,1)^d$, such that for any two (distinct) points $p, q \in [0,1)^d$, with $\ell = \|p-q\|$, there is an ordering $\sigma \in \Pi^+$, and a point $z \in [0,1)^d$, such that*

(i) $p \prec_\sigma q$,

(ii) $(p,z)_\sigma \subseteq \mathrm{ball}\big(p, \varsigma\ell\big)$,

(iii) $(z,q)_\sigma \subseteq \mathrm{ball}\big(q, \varsigma\ell\big)$, *and*

(iv) $z \in \mathrm{ball}\big(p, \varsigma\ell\big)$ *or* $z \in \mathrm{ball}\big(q, \varsigma\ell\big)$.

Furthermore, given such an ordering σ, and two points p, q, one can compute their ordering, according to σ, using $O(d \log \varsigma^{-1})$ arithmetic and bitwise-logical operations.

4.1 Construction

Given a set P of n points in $[0,1)^d$, and parameters $\varepsilon, \vartheta \in (0,1)$, let $\varsigma = \varepsilon/(c \log n)$,

$$M = 4M(\varsigma) = \mathcal{O}(\varsigma^{-d} \log \varsigma^{-1}) = \mathcal{O}\left(\varepsilon^{-d} \log^d n \log \frac{\log n}{\varepsilon}\right),$$

and c be some sufficiently large constant. Next, let $\vartheta' = \vartheta/M$, and let $\Pi^+ = \Pi^+(\varsigma)$ be the set of orderings of Theorem 4.1. For each ordering $\sigma \in \Pi^+$, compute the ϑ'-reliable exact spanner G_σ of P, see Theorem 3.4, according to σ. Let G be the resulting graph by taking the union of G_σ for all $\sigma \in \Pi^+$.

4.2 Analysis

▶ **Theorem 4.2.** *The graph G constructed above is a ϑ-reliable $(1+\varepsilon)$-spanner and has size $\mathcal{O}\Big(\varepsilon^{-7d}\vartheta^{-6} n \log^{7d} n \log^7 \frac{\log n}{\varepsilon}\Big)$.*

Proof. Given a (failure) set $B \subseteq P$, let B^+ be the union of all the harmed sets resulting from B in G_σ, for all $\sigma \in \Pi^+$. We have that $|B^+| \leq (1 + M \cdot \vartheta')\,|B| = (1+\vartheta)\,|B|$.

Consider any two points $p, q \in P \setminus B^+$. By Theorem 4.1, for $\ell = \|p-q\|$, there exists an ordering $\sigma \in \Pi^+$, and a point $z \in [0,1)^d$, such that $(p,z)_\sigma \subseteq \mathrm{ball}(p, \varsigma\ell)$ and $(z,q)_\sigma \subseteq \mathrm{ball}(q, \varsigma\ell)$ (and z is in one of these balls).

By Theorem 3.4, the graph $G_\sigma \setminus B \subseteq G \setminus B$ contains a monotone path π, according to σ, with $h = \mathcal{O}(\log n)$ hops, connecting p to q. Let $p = p_1, \ldots, p_{h+1} = q$ be this path. Observe that there is a unique index i, such that $z \in (p_i, p_{i+1})$. We have the following:

(A) $\forall j \neq i \quad \|p_j - p_{j+1}\| \leq 2\varsigma\ell$.

(B) $\|p_i - p_{i+1}\| \leq \ell + 2\varsigma\ell$.

As such, the total length of π is $\sum_{j=1}^{h} \|p_j - p_{j+1}\| = (1 + 2\varsigma h)\ell \leq (1+\varepsilon)\ell$, as desired, if c is sufficiently large. Namely, G is the desired reliable spanner.

The number of edges of G is

$$M \cdot \mathcal{O}\big((\vartheta')^{-6} n \log n\big) = \mathcal{O}\big(M (M/\vartheta)^6 n \log n\big) = \mathcal{O}\left(\varepsilon^{-7d}\vartheta^{-6} n \log^{7d} n \log^7 \frac{\log n}{\varepsilon}\right). \qquad ◀$$

4.3 Improved constructions

By setting $\varsigma = \varepsilon/c$ in the above construction and applying a more careful analysis we can improve this result, which is stated in the following theorem.

▶ **Theorem 4.3.** *One can construct a ϑ-reliable $(1+\varepsilon)$-spanner with size*

$$\mathcal{O}\left(\varepsilon^{-7d} \log^7 \frac{1}{\varepsilon} \cdot \vartheta^{-6} n \log n (\log\log n)^6\right).$$

Using another construction we are able to obtain a ϑ-reliable $(1+\varepsilon)$-spanner with size $\mathcal{O}\big(\varepsilon^{-d}\vartheta^{-2}n\log n\big)$ if the underlying point set P has polynomially bounded spread, which is optimal. For both of these results see the full version [7].

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