Maximum Matchings and Minimum Blocking Sets in Θ_6-Graphs

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Abstract

Θ_6-graphs are important geometric graphs that have many applications. They are equivalent to Delaunay graphs where empty equilateral triangles with a horizontal edge take the place of empty circles. We investigate lower bounds on the size of maximum matchings in these graphs. The best known lower bound is $(n-1)/3$, where n is the number of vertices of the graph. Babu et al. (2014) conjectured that any Θ_6-graph has a perfect matching (as is true for standard Delaunay graphs). Although this conjecture remains open, we improve the lower bound to $(3n-8)/7$.

We also relate the size of maximum matchings in Θ_6-graphs to the minimum size of a *blocking set*. Every edge of a Θ_6-graph on a point set P corresponds to an empty triangle that contains the endpoints of the edge but no other point of P. A *blocking set* has at least one point in each such triangle. We prove that the size of a maximum matching is at least $\beta(n)/2$ where $\beta(n)$ is the minimum, over all Θ_6-graphs with n vertices, of the minimum size of a blocking set. In the other direction, lower bounds on matchings allow us to show that $\beta(n) \geq 3n/4 - 2$.

1 Introduction

One of the many beautiful properties of Delaunay triangulations is that they always contain a perfect matching, as proved by Dillencourt [10]. This is one example of a structural property of a so-called *proximity graph*. A proximity graph is determined by a set $\mathcal{S}$ of geometric objects in the plane, such as all discs, or all axis-aligned squares. Given such a set $\mathcal{S}$ and a finite point set P, we construct a proximity graph with vertex set P and with an edge (p, q) if there is an object from $\mathcal{S}$ that contains p and q, and no other point of P in its interior. When $\mathcal{S}$ consists of all discs, then we get the Delaunay triangulation.

Our paper is about structural properties of Θ_6-graphs, which are the proximity graphs determined by equilateral triangles with a horizontal edge. More precisely, for any finite point set P, define $G^{\triangle}(P)$ to be the proximity graph of P with respect to *upward* equilateral triangles $\triangle$, define $G^{\bigtriangledown}(P)$ to be the proximity graph of P with respect to *downward* equilateral triangles $\bigtriangledown$, and define $G^{\text{✡}}(P)$, the Θ_6-graph of P, to be their union. In particular, for two points p and q in the plane, we denote by $\triangle(p, q)$ (resp., by $\bigtriangledown(p, q)$) the smallest upward (resp., downward) equilateral triangle that has p and q on its boundary. We say that a triangle is *empty* if it has no points of P in its interior. With these definitions, $G^{\text{✡}}(P)$ has an edge between p and q if and only if $\triangle(p, q)$ is empty or $\bigtriangledown(p, q)$ is empty, in which case we say that the edge (p, q) is *introduced* by $\triangle(p, q)$ or by $\bigtriangledown(p, q)$.

Θ_6-graphs were first introduced by Clarkson [9] and Keil [11] as follows. Place 6 rays emanating from every point $p \in P$ at angles that are multiples of $\pi/3$ from the positive x-

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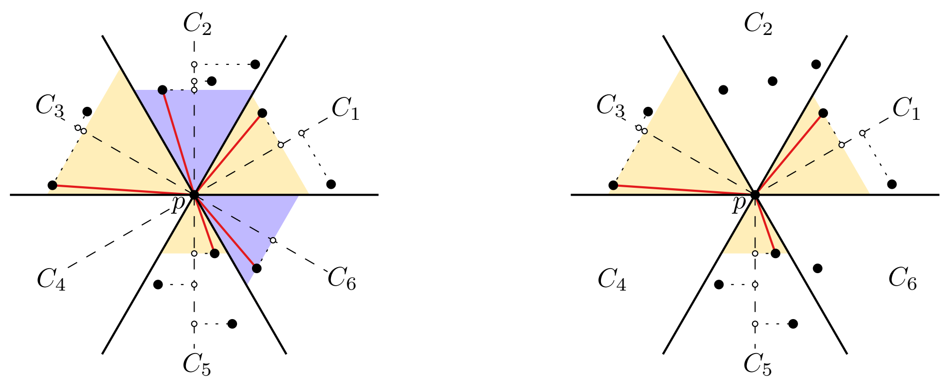
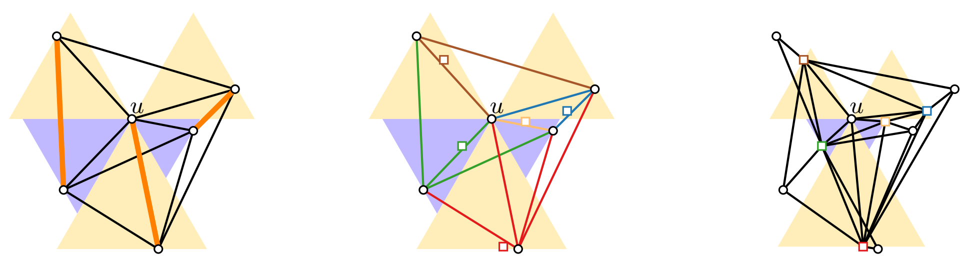


(a) **(b)**

Figure 1 The construction of (a) the Θ_6-graph, and (b) the odd half-Θ_6-graph.



(a) **(b)** **(c)**

Figure 2 A Θ_6-graph on $n = 6$ points with a perfect matching and a blocking set of size 5. (a) A perfect matching. (b) A blocking set B of size $n - 1$. Edges have the same color as their blocking point. (c) $G^{\text{✡}}(P \cup B)$. All edges are incident to a point of B.

axis. These rays partition the plane into 6 cones with apex p, which we label $C_1, \ldots, C_6$ in counterclockwise order starting from the positive x-axis; see Figure 1a. Add an edge from p to the *closest* point in each cone C_i, where the distance between the apex p and a point q in C_i is measured by the Euclidean distance from p to the projection of q on the bisector of C_i as depicted in Figure 1a. It is straight-forward to show both definitions of Θ_6-graphs are equivalent. The edges of $G^{\triangle}(P)$ come from the odd cones, and the edges of $G^{\triangledown}(P)$ come from the even cones, so the graphs $G^{\triangle}(P)$ and $G^{\triangledown}(P)$ are known as "half-Θ_6" graphs.

We explore two conjectures about Θ_6-graphs; see Figure 2 for an example.

▶ **Conjecture 1** (Babu et al. [3]). *Every Θ_6-graph has a perfect matching.*

The best known bound is that every Θ_6-graph on n points has a matching of size at least $\lceil (n-1)/3 \rceil$; see Babu et al. [3]. Our main result is an improvement of this lower bound:

▶ **Theorem 1.** *Every Θ_6-graph has a matching of size at least $(3n - 8)/7$.*

We prove Theorem 1 using a technique that has been used for matchings in planar proximity graphs, namely the Tutte-Berge theorem, which relates the size of a maximum matching in a graph to the number of components of odd cardinality after removing some vertices. In our case, this approach is more complicated because Θ_6-graphs are not necessarily planar.

Our second main result relates the size of matchings to the size of *blocking* sets of proximity graphs, which were introduced by Aronov et al. [1]. For a proximity graph $G(P)$

defined in terms of a set of objects $\mathcal{S}$, a set B of points *blocks* $G(P)$ if B has a point inside any object from $\mathcal{S}$ that contains exactly two points of P, i.e., the set B destroys all the edges of $G(P)$, or equivalently, $G(P \cup B)$ has no edges between vertices in P. Refer to Figure 2.

For a set of points P, let $\beta(G^{\text{✡}}(P))$ be the minimum size of a blocking set of $G^{\text{✡}}(P)$. Let $\beta(n)$ be the minimum, over all point sets P of size n, of $\beta(G^{\text{✡}}(P))$. It is known that $\beta(n) \geq \lceil (n-1)/2 \rceil$ since that already holds for $G^{\triangle}$-graphs [8]. Let $\mu(n)$ be the minimum, over all point sets P of size n, of the size of a maximum matching in $G^{\text{✡}}(P)$. Conjecture 1 can hence be restated as $\mu(n) \geq \lceil (n-1)/2 \rceil$. We relate the parameters μ and β as follows:

▶ **Theorem 2.** *(a) Every Θ_6-graph has a matching of size $\beta(n)/2$, i.e., $\mu(n) \geq \beta(n)/2$.*
(b) If $\mu(n) \geq cn + d$ for some constants c, d, then $\beta(n) \geq (cn + d)/(1 - c)$.

Theorem 2 has two consequences. The first is that Theorem 1 implies the following.

▶ **Corollary 3.** *$\beta(n) \geq 3n/4 - 2$.*

The second consequence is that Conjecture 1 is equivalent to the following conjecture.

▶ **Conjecture 2.** *$\beta(n) \geq n - 1$.*

Some proofs are sketched, the full proofs can be found in the full version of the paper [6].

2 Preliminaries

We assume that points are in general position and that no line passing through two points of P makes an angle of 0°, 60°, or 120° with the horizontal.

▶ **Lemma 4** (Babu et al. [3]). *Let P be a set of points in the plane, and let p and q be any two points in P. There is a path between p and q in $G^{\triangle}(P)$ that lies entirely in $\triangle(p, q)$. Moreover, the triangles that introduce the edges of this path also lie entirely in $\triangle(p, q)$. Analogous statements hold for $G^{\triangledown}(P)$ and $\triangledown(p, q)$.*

The next lemma has been proved in the general setting of convex-distance Delaunay graphs. We state the result for our special case. For two points p and q in the plane, define the weight function $w^{\triangle}(p, q)$ to be the scaling factor, relative to the unit triangle, of $\triangle(p, q)$.

▶ **Lemma 5** (Aurenhammer and Paulini [2], Theorem 4). *The minimum spanning tree of points P with respect to the weight function $w^{\triangle}(p, q)$ is contained in $G^{\triangle}(P)$.*

A consequence of Lemma 5 (as noted by Aurenhammer and Paulini in their more general setting) is that the minimum spanning tree of points P with respect to the weight function $w^{\triangle}(p, q)$ is contained in both $G^{\triangle}(P)$ and $G^{\triangledown}(P)$, because $w^{\triangle}(p, q) = w^{\triangledown}(p, q)$.

The Tutte-Berge Matching Theorem. Let G be a graph and let S be an arbitrary subset of vertices of G. Removing S will split G into a number $\mathrm{comp}(G \setminus S)$ of connected components. Let $\mathrm{odd}(G \setminus S)$ be the number of odd components of $G \setminus S$. In 1947, Tutte [12] characterized graphs that have a perfect matching as exactly those graphs that have at most $|S|$ odd components for any subset S. In 1957, Berge [5] extended this result to a formula (today known as Tutte-Berge formula) for the size of maximum matchings in graphs. The following is an alternate way of stating this formula in terms of the number of unmatched vertices, i.e., vertices that are not matched by the matching.

▶ **Theorem 6** (Tutte-Berge formula; Berge [5]). *The number of unmatched vertices of a maximum matching in G is equal to the maximum over subsets $S \subseteq V$ of $\mathrm{odd}(G \setminus S) - |S|$.*

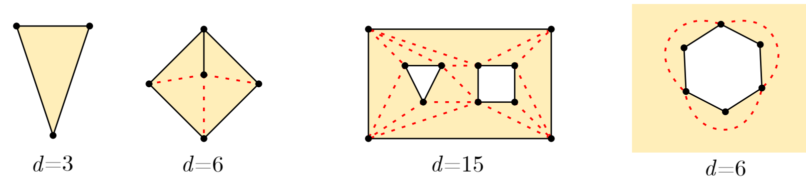


Figure 3 The notion of degree of a face.

We will use this formula in our proofs of Theorems 1 and 2. In fact, as in Dillencourt's proof [10] that Delaunay graphs have perfect matchings, we will find an upper bound on $\mathrm{comp}(G \setminus S) - |S|$, i.e., we establish a bound on the *toughness* of the graph [4]. We define the *degree* of a face as the number of triangles in a triangulation of it plus 2; see Figure 3.

3 Bounding the Size of a Matching

Let P be a set of n points in the plane. We will prove Theorem 1—that $G^{\text{✡}}(P)$ contains a matching of size at least $(3n-8)/7$. It is known that all interior faces of $G^{\triangle}(P)$ and $G^{\triangledown}(P)$ are triangles [3], but their outer face might be non-convex, so we add a set $A = \{a_1, \ldots, a_6\}$ of *surrounding points* near the corners of the smallest upward and downward equilateral triangle $T^{\triangle}$ and $T^{\triangledown}$, containing all points of P; see Figure 4.

Fix a set S for which we want to bound $\mathrm{comp}(G^{\text{✡}}(P) \setminus S) - |S|$, and define $S_A = S \cup A$. Pick an arbitrary representative point from every connected component of $G^{\text{✡}}(P) \setminus S$, and let Q be the set of these points, so $|Q| = \mathrm{comp}(G^{\text{✡}}(P) \setminus S)$.

Define $G_A^{\triangle} = G^{\triangle}(P \cup A)$ and consider its subgraph $G_A^{\triangle}[S_A]$. Note that the outer face of both $G_A^{\triangle}$ and $G_A^{\triangle}[S_A]$ is the hexagon formed by A; we add three graph edges to triangulate the outer face, so every face of $G_A^{\triangle}$ is a triangle.

Let $f_d^{\triangle}$ be the number of faces of degree d in $G_A^{\triangle}[S_A]$ that contain a point of Q. Let $f_{4+}^{\triangle} = \sum_{d \geq 4} f_d^{\triangle}$. As all faces of $G_A^{\triangle}$ are triangles, we know from Dillencourt [10, Lemma 3.4] that every face of $G_A^{\triangle}[S_A]$ contains at most one component, so at most one point of Q. Thus,

$$|Q| = f_3^{\triangle} + f_{4+}^{\triangle} \qquad \text{and similarly} \qquad |Q| = f_3^{\triangledown} + f_{4+}^{\triangledown}, \tag{1}$$

where $f_d^{\triangledown}$ is defined in a symmetric manner on graph $G_A^{\triangledown}[S_A]$.

Let $\mathcal{F}_d$ be the set of faces of degree d in $G_A^{\triangle}$ and observe that, since no point of Q appears in the four triangles outside the hexagon of A, we have $f_3^{\triangle} \leq |\mathcal{F}_3| - 4$. An easy counting

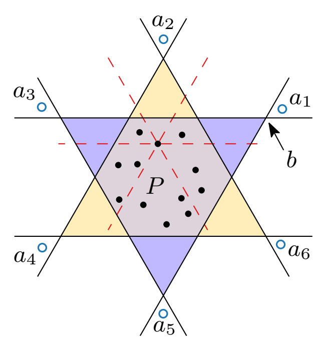


Figure 4 Augmentation of P: the shaded regions are $T^{\triangle}$ and $T^{\triangledown}$, and $A = \{a_1, \ldots, a_6\}$.

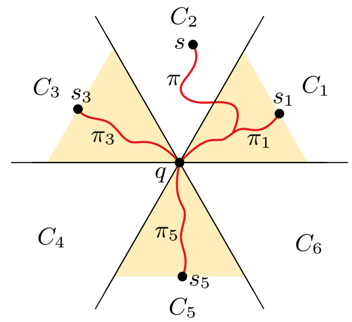


Figure 5 Illustration for the proof of Lemma 7.

argument (also used by Biedl et al. [7]) shows that $\sum_{d\geq 3}(d-2)|\mathcal{F}_d(G)| = 2|V|-4$. Thus,

$$\begin{aligned} f_3^{\triangle} + 2f_{4+}^{\triangle} \leq \sum_{d\geq 3}(d-2)f_d^{\triangle} &\leq \sum_{d\geq 3}(d-2)|\mathcal{F}_d| - 4 \\ &\leq 2|V(G_A^{\triangle})| - 4 - 4 = 2|S| + 2|A| - 8 = 2|S| + 4, \qquad (2) \end{aligned}$$

and similarly $f_3^{\triangledown} + 2f_{4+}^{\triangledown} \leq 2|S|+4$. The crucial insight for getting an improved matching bound is that no component can reside inside a face of degree 3 in both $G^{\triangle}$ and $G^{\triangledown}$.

▶ **Lemma 7.** *We have $f_3^{\triangle} \leq f_{4+}^{\triangledown}$ and $f_3^{\triangledown} \leq f_{4+}^{\triangle}$.*

Proof sketch. Take any point $q \in Q$. Find the shortest path π in the minimum-weight spanning tree T of $P \cup A$ that connects q to some point $s \in S_A$. Assume w.l.o.g. that s is in cone C_2; see Figure 5. Let π_1, π_3, π_5 be the paths from q to the points s_1, s_3, s_5 of S_A that are closest to q in cones C_1, C_3, C_5 that lie fully in $\triangle(q, s_i)$, respectively (exists by Lemma 4). Then, no interior vertex of π, π_1, π_3, π_5 is in S_A. Hence, s, s_1, s_3, s_5 belong to the boundary of the same face $F^{\triangle}$ of $G^{\triangle}[S_A]$ that contains q, so $F^{\triangle}$ has degree at least 4. ◀

Now we have the tools to prove an upper bound on the toughness of a Θ_6-graph.

▶ **Lemma 8.** *For any $S \subseteq P$, we have* $\mathrm{comp}(G^{\davidsstar}(P) \setminus S) - |S| \leq (n+16)/7$.

Proof. Recall that we fixed a set Q of points in $P \setminus S$ with $|Q| = \mathrm{comp}(G^{\davidsstar}(P) \setminus S)$. So $n = |P| \geq |S| + |Q|$. Combining this with the above inequalities, we get

$$\begin{aligned} 7\left(\mathrm{comp}(G^{\davidsstar}(P) \setminus S) - |S|\right) &\leq 7|Q| - 7|S| + (n - |Q| - |S|) = n + 3|Q| + 3|Q| - 8|S| \\ &= n + 3\left(f_3^{\triangle} + f_{4+}^{\triangle}\right) + 3\left(f_3^{\triangledown} + f_{4+}^{\triangledown}\right) - 8|S| \qquad \text{(by (1))} \\ &\leq n + 2f_3^{\triangle} + 4f_{4+}^{\triangle} + 2f_3^{\triangledown} + 4f_{4+}^{\triangledown} - 8|S| \qquad \text{(by Lemma 7)} \\ &\leq n + (4|S|+8) + (4|S|+8) - 8|S| = n + 16. \qquad \text{(by (2))} \end{aligned}$$ ◀

Therefore, $\mathrm{odd}(G^{\davidsstar}(P) \setminus S) - |S| \leq \mathrm{comp}(G^{\davidsstar}(P) \setminus S) - |S| \leq (n+16)/7$. In consequence of the Tutte-Berge formula, therefore any maximum matching M of $G^{\davidsstar}(P)$ has at least $(6n-16)/7$ matched vertices and $|M| \geq (3n-8)/7$. This completes the proof of Theorem 1.

4 The Relationship Between Blocking Sets and Matchings

In this section, we prove Theorem 2—that a lower bound on the blocking size function $\beta(n)$ implies a lower bound on the size $\mu(n)$ of a maximum matching, and vice versa.

▶ **Lemma 9.** *For any $n \geq 1$, we have $\beta(n+1) \leq \beta(n) + 1$.*

Proof sketch. Consider a set P with n points such that $\beta(n) = \beta(G^{\text{✡}}(P))$. Consider the points a_1 and b depicted in Figure 4. We can block $G^{\text{✡}}(P \cup \{a_1\})$ by using a minimum blocking set B of $G^{\text{✡}}(P)$ and adding b to it, so $\beta(n+1) \leq \beta(G^{\text{✡}}(P \cup \{a_1\})) \leq \beta(n)+1$. ◀

Since $\beta(1) = 0$, this lemma also shows that $\beta(n) \leq n-1$, i.e., that Conjecture 2 is tight.

▶ **Theorem 2.** *(a) Every Θ_6-graph has a matching of size $\beta(n)/2$, i.e., $\mu(n) \geq \beta(n)/2$.*

Proof. Fix a point set P, an arbitrary set $S \subseteq P$, one representative point in each connected component of $G^{\text{✡}}(P) \setminus S$, and let Q be the set of these points. Let (q_1, q_2) be an edge in $G^{\text{✡}}(Q)$ introduced by $\triangle(q_1, q_2)$. By Lemma 4, there is a path π between q_1 and q_2 in $G^{\triangle}(P)$ that is fully contained in $\triangle(q_1, q_2)$. Since q_1 and q_2 belong to different components of $G^{\text{✡}}(P) \setminus S$, at least one point of π belongs to S. Thus, S blocks $G^{\text{✡}}(Q)$, and $|S| \geq \beta(|Q|)$. Further, $\beta(n) \leq \beta(|Q|) + n - |Q|$ by Lemma 9 since $|Q| \leq n$. By Theorem 6, it follows that

$$\mu(G^{\text{✡}}(P)) \geq \frac{n - (|Q| - |S|)}{2} \geq \frac{n - (|Q| - \beta(|Q|))}{2} \geq \frac{n - (n - \beta(n))}{2} = \frac{\beta(n)}{2}. \quad ◀$$

In particular, if $\beta(n) \geq n-1$, then $\mu(n) \geq \beta(n)/2 \geq (n-1)/2$, so by integrality $\mu(n) \geq \lceil (n-1)/2 \rceil$. In other words, Conjecture 2 implies Conjecture 1.

▶ **Theorem 2.** *(b) If $\mu(n) \geq cn + d$ for some constants c, d, then $\beta(n) \geq (cn+d)/(1-c)$.*

Proof. Let P be a set of n points such that $\beta(G^{\text{✡}}(P)) = \beta(n) = b$, and let B be a minimum blocking set of $G^{\text{✡}}(P)$ of size b. Let M be a matching of size at least $\mu(b+n) \geq cb + cn + d$ in $G^{\text{✡}}(P \cup B)$. Since P is an independent set in $G^{\text{✡}}(P \cup B)$, it can contain at most one endpoint of each edge in M, as well as all unmatched points, so

$$n = |P| \leq |M| + (n + b - 2|M|) \leq n + b - (cb + cn + d).$$

Solving for b gives $\beta(n) = b \geq (cn+d)/(1-c)$. ◀

In particular, if Conjecture 1 holds, then $\mu(n) \geq (n-1)/2$. Hence, $c = -d = 1/2$, so $\beta(n) \geq 2(n-1)/2 = n-1$ and Conjecture 2 holds. So Conjecture 1 implies Conjecture 2. As a second consequence, we know that $(3n-8)/7$ is a valid lower bound on $\mu(n)$ by Theorem 1, therefore (with $c = 3/7$) we have $\beta(n) \geq 7/4 \cdot (3n-8)/7 = 3n/4 - 2$, proving Corollary 3.

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