Simultaneous Representation of Proper and Unit Interval Graphs

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Abstract

A simultaneous representation of graphs $G_1, \dots, G_k$ consists of a (geometric) intersection representation R_i for each graph G_i such that for each pair of graphs G_i and G_j the representations R_i and R_j are *compatible* in the sense that vertices shared by G_i and G_j are represented by the same geometric object in R_i and in R_j. An important special case is the *sunflower case*, where we require that $G_i \cap G_j$ yields the same *shared graph* S for each $i \neq j$. While the existence of simultaneous interval representations for $k = 2$ can be tested efficiently, testing it for non-sunflower graphs with k not fixed is NP-complete. We give efficient algorithms for testing the existence of simultaneous proper and unit interval representations for sunflower graphs with k not fixed.

1 Introduction

A fundamental problem in the area of intersection graphs is the *recognition* problem, where the task is to decide whether a given graph G admits a particular type of (geometric) intersection representation. The simultaneous representation problem is a generalization of the recognition problem which asks for a *simultaneous graph* $\mathcal{G} = (G_1, \dots, G_k)$ whether it admits a simultaneous geometric representation $\mathcal{R} = (R_1, \dots, R_k)$.

Simultaneous representations have first been studied in the context of graph embeddings where the goal is to embed each simultaneous graph without edge crossings while any shared vertices have the same coordinates in all embeddings; see [1] for a survey. The notion of simultaneous representation of general intersection graph classes was introduced by Jampani and Lubiw [9]. They gave an $O(n^2 \log n)$ recognition algorithm for simultaneous interval graphs with $k = 2$ [8]. Bläsius and Rutter later improved the running time to linear [2]. Bok and Jedličková very recently showed that recognizing simultaneous non-sunflower interval graphs with k not fixed is NP-complete [3]. The problem is open in the sunflower case.

Contribution. We settle these problems with k not fixed for simultaneous proper and unit interval graphs – those graphs with an interval representation where no interval properly contains another and where all intervals have unit length, respectively. For the sunflower case, we provide efficient recognition algorithms. The running time for proper interval graphs is linear, while for the unit case it is $\mathcal{O}(|V| \cdot |E|)$ where V and E are the set of vertices and edges in the union of the sunflower graphs, respectively. For the non-sunflower case, we prove NP-completeness. The reductions are similar to the simultaneous independent work of Bok and Jedličková for simultaneous interval graphs [3].

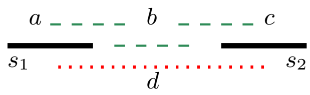

Figure 1 A simultaneous proper interval representation of a sunflower graph $\mathcal{G} = (P_5, P_3)$ without simultaneous unit interval representation (P_5 green dashed, P_3 red dotted, $P_5 \cap P_3$ black bold).

2 Preliminaries

All graphs in this paper are undirected. An *interval representation* $R = \{I_v \mid v \in V\}$ of a graph $G = (V, E)$ associates with each vertex $v \in V$ an interval $I_v = [x, y] \subset \mathbb{R}$ such that for each pair of vertices $u, v \in V$ we have $I_u \cap I_v \neq \emptyset \Leftrightarrow uv \in E$. An interval representation R is *proper* if no interval properly contains another one, and it is *unit* if all intervals have length 1. A graph is a *(proper/unit) interval graph* if and only if it admits a (proper/unit) interval representation. It is well-known that proper and unit interval graphs are the same graph class. However, the simultaneous unit interval graphs are a strict subclass of the simultaneous proper interval graphs; see Figure 1.

We use the well-known characterization of proper interval graphs using *straight enumerations* [6]. Two adjacent vertices $u, v \in V$ are *indistinguishable* if we have $N[u] = N[v]$ where $N[u] = \{v : uv \in E(H)\} \cup \{u\}$ is the closed neighborhood. Being indistinguishable is an equivalence relation and we call the equivalence classes *blocks* of G. Two blocks B, B' are *adjacent* if and only if $uv \in E$ for (any) $u \in B$ and $v \in B'$. A linear ordering σ of the blocks of G is a *straight enumeration* of G if for every block, the block and its adjacent blocks are consecutive in σ. A proper interval representation R defines a straight enumeration $\sigma(R)$ by ordering the intervals by their starting points and grouping together the blocks. Conversely, for each straight enumeration σ, there exists a corresponding representation R with $\sigma = \sigma(R)$ [6]. A *fine enumeration* of a graph H is a linear ordering η of $V(H)$ such that for $u \in V(H)$ the closed neighborhood $N[u]$ is consecutive in η.

▶ **Proposition 2.1** ([11, 6, 7])**.** *For a graph G the following statements are equivalent: (i) G is a proper interval graph, (ii) G has a straight enumeration, (iii) G has a fine enumeration. Also, for a connected proper interval graph its straight enumeration is unique up to reversal.*

In the following we only consider sunflower graphs $\mathcal{G} = (G_1, \dots, G_k)$ with shared graph S. Note that it is necessary that S is an induced subgraph of each input graph G_i. Also note that $\mathcal{G}$ admits a simultaneous (proper/unit) interval representation if and only if each component of its union graph $\bigcup_{i=1}^{k} G_i$ does. We hence restrict our attention to sunflower graphs that are *connected* in the sense that their union graph is connected.

3 Sunflower Proper Interval Graphs

Let $\mathcal{G} = (G_1, \dots, G_k)$ be a sunflower graph with shared graph $S = (V_S, E_S)$. By Proposition 2.1 each G_i has at least one fine enumeration. If there are fine enumerations $\sigma_1, \dots, \sigma_k$ of $G_1, \dots, G_k$ that coincide on V_S, then they induce a fine enumeration σ_S of S. We can then find a proper interval representation of S corresponding to σ_S that can be extended to proper interval representations of $G_1, \dots, G_k$ in linear time [10]. Otherwise there is no simultaneous proper interval representation. Using PQ-trees [5, 4], the existence of such an ordering σ_S can be tested in linear time.

▶ **Theorem 3.1.** *Given a sunflower graph $\mathcal{G} = (G_1, \dots, G_k)$, it can be tested in linear time whether $\mathcal{G}$ admits a simultaneous proper interval representation.*

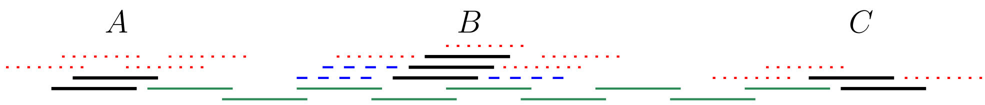

Figure 2 Simultaneous proper interval representation of G_1 (green solid), G_2 (red dotted), G_3 (blue dashed) with shared graph S (black bold). S has three blocks A, B, C. We denote the component of G_i containing a block D by C_D^i. C_A^2, C_B^2, C_B^3, C_C^2 are loose. C_A^2 is independent. (C_B^2, C_B^3) is a reversible part. (C_C^2) is not a reversible part, since C_C^1 is aligned at C and not loose.

Next we characterize all simultaneous proper interval representations of a sunflower graph. Let $\mathcal{G} = (G_1, \ldots, G_k)$ be a sunflower graph with shared graph $S = (V_S, E_S)$ and for each $G_i \in \mathcal{G}$ let σ_i be a straight enumeration of G_i. We call the tuple $(\sigma_1, \ldots, \sigma_k)$ a *simultaneous enumeration* if for any $i, j \in \{1, \ldots, k\}$ and $u, v \in V_S$ the blocks $B_i(u)$, $B_i(v)$ and $B_j(u)$, $B_j(v)$ of G_i and G_j containing u, v are not ordered differently by σ_i and σ_j, i.e., we do not have $(B_i(u), B_i(v)) \in \sigma_i$ and $(B_j(v), B_j(u)) \in \sigma_j$ or vice versa.

▶ **Theorem 3.2.** *Let $\mathcal{G} = (G_1, \ldots, G_k)$ be a sunflower graph. There exists a simultaneous proper interval representation $\mathcal{R} = (R_1, \ldots, R_k)$ of $\mathcal{G}$ if and only if there is a simultaneous enumeration $(\sigma_1, \ldots, \sigma_k)$ of $\mathcal{G}$. If $(\sigma_1, \ldots, \sigma_k)$ exists, there also exists a simultaneous proper interval representation $\mathcal{R} = (R_1, \ldots, R_k)$ with $(\sigma(R_1), \ldots, \sigma(R_k)) = (\sigma_1, \ldots, \sigma_k)$.*

It turns out there is a unique straight enumeration of S induced by all simultaneous proper interval representations of $\mathcal{G}$ (up to reversal) if $\mathcal{G}$ is connected. For the following definitions see Figure 2. Let C be a component of a graph G in $\mathcal{G}$. We call C *loose* if all shared vertices in C are in the same block of S. Reversal of loose components is the only "degree of freedom" among simultaneous enumerations, besides full reversal. We say two vertices $u, v \in V_S$ *align* C if they are in different blocks of C. We call C *independent* if it is loose and not aligned by any two vertices of S.

We say C *is aligned at* a block B of S if it is aligned by two vertices u, v in B. Any two components aligned at the same block can not be reversed independently. For each block B of S, let $\mathcal{C}(B)$ be the connected components among graphs in $\mathcal{G}$ aligned at B. If all components in $\mathcal{C}(B)$ are loose, we call it a *reversible part*. Note that a reversible part contains at most one component of each graph G_i. Let $(\sigma_1, \ldots, \sigma_k)$ and $(\sigma_1', \ldots, \sigma_k')$ be tuples of straight enumerations of $G_1, \ldots, G_k$. We say $(\sigma_1', \ldots, \sigma_k')$ is *obtained from* $(\sigma_1, \ldots, \sigma_k)$ *by reversing* reversible part $\mathcal{C}(B)$ if $\sigma_1', \ldots, \sigma_k'$ are obtained by reversal of all components in $\mathcal{C}(B)$. We characterize the simultaneous enumerations of $\mathcal{G}$ as follows.

▶ **Theorem 3.3.** *Let $\mathcal{G} = (G_1, \ldots, G_k)$ be a connected sunflower graph with simultaneous enumeration ρ. Then ρ' is a simultaneous enumeration of $\mathcal{G}$ if and only if ρ' can be obtained from ρ or its reversal ρ^r by reversing independent components and reversible parts.*

4 Sunflower Unit Interval Graphs

We now characterize for a sunflower graph $\mathcal{G} = (G_1, \ldots, G_k)$ with shared graph S the simultaneous enumerations $(\zeta_1, \ldots, \zeta_k)$ that can be *realized* by a simultaneous unit interval representation $(R_1, \ldots, R_k)$, in the sense that $\sigma(R_i) = \zeta_i$ for $i \in \{1, \ldots, k\}$. For $i \in \{1, \ldots, k\}$ let $(V_i, E_i) = G_i$. Let further $V = V_1 \cup \cdots \cup V_k$. For a straight enumeration η of some graph H we say for $u, v \in V(H)$ that $u <_\eta v$ if u is in a block before v, and we say $u \leq_\eta v$ if $u = v$ or $u <_\eta v$. We call $\leq_\eta$ the *partial order on $V(H)$ corresponding to η*. Note that for distinct u, v in the same block we have neither $u >_\eta v$ nor $u \leq_\eta v$. For convenience, we write $u \leq_i v$ and $u <_i v$ instead of $u \leq_{\zeta_i} v$ and $u <_{\zeta_i} v$, respectively.

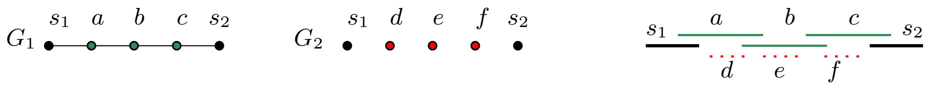

Figure 3 A sunflower graph $\mathcal{G} = (G_1, G_2)$ with shared vertices s_1, s_2. In the corresponding simultaneous enumeration ζ we have the (s_1, s_2)-chain $C = (s_1, a, b, c, s_2)$ and the (s_1, s_2)-bar $B = (s_1, d, e, f, s_2)$, both of size 5. Hence, $\mathcal{G}$ has conflict (C, B) for ζ.

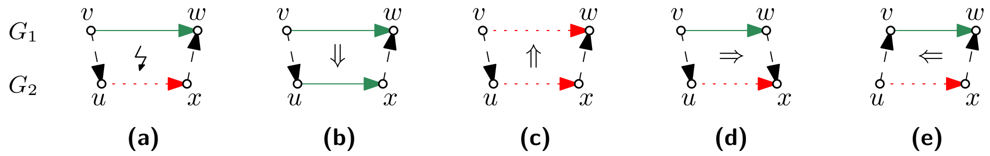

Figure 4 Two graphs G_1, G_2 with $V_1 = \{v, w\}$, $V_2 = \{u, x\}$, $u \leq_\eta x$, and $v \leq_\eta w$. In Figure 4a we have a forbidden configuration with (i) $vw \in E_1$, (ii) $ux \notin E_2$, (iii) $v \leq_\eta u$, and (iv) $x \leq_\eta w$. If three of these four conditions are met, we can conclude that the remaining one is false. Namely, in Figure 4b, 4c, 4d and 4e, we conclude $ux \in E_2$, $vw \notin E_1$, $w <_\eta x$, and $u <_\eta v$, respectively. We use arrows to represent a partial order between two vertices. We draw them green solid if they are adjacent, red dotted if they are non-adjacent in some graph G_i, and black dashed otherwise.

Let $u, v \in V_S$ with $u \neq v$. A (u, v)-*chain* of *size* m in (G_i, ζ_i) is a sequence $(u = c_1, \dots, c_m = v)$ of vertices in V_i with $c_1 <_i \dots <_i c_m$ that corresponds to a path in G_i. A (u, v)-*bar* between u and v of *size* m in (G_i, ζ_i) is a sequence $(u = b_1, \dots, b_m = v)$ of vertices in V_i with $b_1 <_i \dots <_i b_m$ that corresponds to an independent set in G_i; see Figure 3.

If there is a (u, v)-chain C in G_i of size $\ell \geq 2$ and a (u, v)-bar B in (G_j, ζ_j) of size at least ℓ, then we say that (C, B) is a *(chain-bar-)conflict* and that $\mathcal{G}$ *has conflict* (C, B) *for* ζ. Note that one can reduce the size of a (u, v)-bar by removing intervals between u, v. Thus, we can always assume that in a conflict, we have a bar and a chain of the same size $\ell \geq 2$.

Assume $\mathcal{G}$ has a simultaneous unit interval representation realizing ζ. If a graph $G \in \mathcal{G}$ has a (u, v)-chain of size $\ell \geq 2$, then I_u, I_v have a distance smaller than $\ell - 2$. On the other hand, if a graph $G \in \mathcal{G}$ has a (u, v)-bar of size ℓ, then I_u, I_v have a distance greater than $\ell - 2$. Hence, sunflower graph $\mathcal{G}$ has no conflict. The absence of conflicts is not only necessary, but also sufficient.

▶ **Theorem 4.1.** *A sunflower graph $\mathcal{G}$ with simultaneous enumeration ζ has a simultaneous unit interval representation that realizes ζ if and only if it has no conflict for ζ.*

Proof Sketch. Let $\alpha^\star$ be the union of the partial orders on $V_1, \dots, V_k$ corresponding to $\zeta_1, \dots, \zeta_k$. We set α to be the transitive closure of $\alpha^\star$, meaning α is the partial order on V induced by ζ. After identifying certain "indistinguishable" vertices of V, we can assume that α is a linear ordering on $V_1, \dots, V_k$. Assuming there is no conflict, we then construct a simultaneous unit interval representation R. To this end, we first extend α to a linear ordering on V and thus of the interval starting points. Afterwards, we decide for every pair u, v of vertices from different graphs whether I_u, I_v intersect to obtain an order of the interval end points. Note that each partial order $\alpha_{|V_i}$ already is a fine enumeration of G_i.

All necessary extensions of α and decisions for adjacencies between vertices of different graphs arise from one *forbidden configuration*; see Figure 4a. We first go from right to left and extend α according to Figure 4e. In that run only necessary extensions are made. The key idea in that run is that the extensions of α correspond to extensions of pairs of chains and bars of equal size with a shared end to the right. If the forbidden configuration is obtained, then such a chain-bar pair also shares the second end and therefore yields a conflict. With this preparation, we can then go from left to right and greedily extend α to a linear ordering τ that respects the implication of Figure 4e. As such τ avoids the forbidden configuration. We finally use τ to decide adjacency for every pair of vertices according to Figure 4b and thereby still avoiding the forbidden configuration. We obtain a graph H that has $G_1, \ldots, G_k$ as induced subgraphs and for which τ is a fine enumeration. By Proposition 2.1 H is a proper and thus a unit interval graph. A unit interval representation of H induces a simultaneous unit interval representation of $\mathcal{G} = (G_1, \ldots, G_k)$. ◀

We now give a recognition algorithm for sunflower unit interval graphs. By Theorem 3.1 we obtain a simultaneous enumeration ζ of $\mathcal{G}$, unless $\mathcal{G}$ is not even a simultaneous proper interval graph. By Theorem 4.1 we need to decide if $\mathcal{G}$ has a simultaneous enumeration η without conflicts. By Theorem 3.3, if it exists, η results from ζ by reversing reversible parts and independent components. We formulate this as a 2-SAT formula with a variable for each reversible part and for each independent component that encodes its orientation.

For each pair of shared vertices u, v we formulate clauses that exclude conflicts for u,v. The minimal (u, v)-chains for G_i are independent of reversals. The size of a largest (u, v)-bar in G_i only depends on the orientations of the connected components C and D containing u and v, respectively, while components in-between always contribute their maximum independent set regardless of whether they are reversed. For each of the at most four relevant combinations of orientations we check whether it produces a conflict. In that case we add a clause that forbids that combination (note that the orientations of C and D are determined by one reversible part or independent component each, if they are loose at all). The 2-SAT formula $\mathcal{F}$ contains these clauses for all shared vertex pairs and all graphs G_i. By construction $\mathcal{F}$ has a solution if and only if $\mathcal{G}$ is a simultaneous unit interval graph.

▶ **Theorem 4.2.** *Given a sunflower graph $\mathcal{G} = (G_1, \ldots, G_k)$, we can decide in $O(|V| \cdot |E|)$ time, whether $\mathcal{G}$ is a simultaneous unit interval graph, where $(V, E) = G_1 \cup \cdots \cup G_k$.*

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