Approximate strong edge-colouring of unit disk graphs

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Abstract

We show that the strong chromatic index of unit disk graphs is efficiently 6-approximable. This improves on 8-approximability as shown by Barrett, Istrate, Kumar, Marathe, Thite, and Thulasidasan (2006). We also show that strong edge-6-colourability is NP-complete for the class of unit disk graphs. Thus there is no polynomial-time $(7/6 - \varepsilon)$-approximation unless P=NP.

1 Introduction

A *strong edge-k-colouring* is a partition of the edges of a graph into k parts so that each part induces a matching. The *strong chromatic index* is the least k for which the graph admits a strong edge-k-colouring. If the vertices of the graph represent communicating nodes, say, in a wireless network, then an optimal strong edge-colouring may represent an optimal discrete assignment of frequencies to transmissions in the network so as to avoid both primary and secondary interference [20, 18, 1]. It is then relevant to model the network geometrically, i.e. as a *unit disk graph* [10]. Our interest is in approximative algorithmic aspects of strong edge-colouring in this model. This was considered by Barrett, Istrate, Kumar, Marathe, Thite, and Thulasidasan [1] who showed that the strong chromatic index of unit disk graphs is efficiently 8-approximable. We revisit the problem and make some further advances.

- We prove efficient 6-approximability.
- We prove efficient online 8-approximability.
- We show impossibility of efficient $(7/6 - \varepsilon)$-approximation unless P=NP.

It is $\exists\mathbb{R}$-complete to decide if a given graph has an embedding as a unit disk graph [12], but both of the approximation algorithms we use are *robust* in the sense that they efficiently output a valid strong edge-colouring upon the input of any abstract graph. Our contribution is to prove that they are guaranteed to output a colouring with good approximation ratio upon the input of a unit disk graph (regardless of any embedding).

Our work parallels and contrasts with work on the chromatic number of unit disk graphs, for which the best approximation ratio known has remained 3 since 1991 [19]. Before stating our main results in detail in Section 3 below, we first review some background material.

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2 Preliminaries

In this section, we highlight some graph theoretic notation, concepts and observations that are relevant to our study. For other standard background, consult e.g. [7]. Given a graph $G = (V, E)$, the *minimum degree*, *clique number*, *chromatic number* and *maximum degree* of G are denoted by $\delta(G)$, $\omega(G)$, $\chi(G)$ and $\Delta(G)$, respectively. The *degeneracy* of G is defined as $\delta^*(G) = \max\{\delta(H) \mid H \subseteq G\}$ and G is called *k-degenerate* if $\delta^*(G) \leq k$. A simple but useful set of inequalities for graph colouring is as follows. For any graph G,

$$\omega(G) \leq \chi(G) \leq \delta^*(G) + 1 \leq \Delta(G) + 1. \tag{1}$$

Note that the second inequality in (1) is algorithmic, in the sense that it follows from the use of an efficient greedy algorithm that always assigns the least available colour, provided we consider the vertices one by one in a suitable order, namely, according to degeneracy. Moreover, a greedy algorithm taking any ordering uses at most $\Delta(G) + 1$ colours.

The *line graph* of G is denoted $L(G)$. The *square* G^2 of G is the graph formed from G by adding all edges between pairs of vertices that are connected by a 2-edge path in G. The *strong chromatic index* of G (as defined above) is denoted $\chi'_2(G)$. Note that $\chi'_2(G) = \chi(L(G)^2)$. The *strong clique number* $\omega'_2(G)$ of G is $\omega(L(G)^2)$. Obviously, (1) implies that

$$\omega'_2(G) \leq \chi'_2(G) \leq \delta^*(L(G)^2) + 1 \leq \Delta(L(G)^2) + 1. \tag{2}$$

It is worth reiterating that the following greedy algorithm efficiently generates a strong edge-$(\delta^*(L(G)^2) + 1)$-colouring: order the edges of G by repeatedly removing from G an edge e for which $\deg_{L(G)^2}(e)$ is lowest, and then colour the edges sequentially according to the *reverse* of this ordering, at each step assigning as a colour the least positive integer that does not conflict with previously coloured edges. Again similarly, with an arbitrary ordering of the edges the greedy algorithm produces a strong edge-$(\Delta(L(G)^2)+1)$-colouring. Our main results will then follow from (2) by suitable bounds on $\delta^*(L(G)^2)$ and $\Delta(L(G)^2)$.

The strong chromatic index is a well-studied parameter in graph theory. Most notably, Erdős and Nešetřil conjectured in the 1980s that $\chi'_2(G) \leq 1.25\Delta(G)^2$ for all graphs G [8]. About a decade later, Molloy and Reed [17] proved the existence of some minuscule but fixed $\varepsilon > 0$ such that $\chi'_2(G) \leq (2 - \varepsilon)\Delta(G)^2$ for all graphs G. Recently there have been improvements [3, 2] and extensions [11, 6], but all rely on Molloy and Reed's original approach, a reduction to a Ramsey-type colouring result. The conjecture remains wide open.

A graph $G = (V, E)$ is said to be a *unit disk graph* if there exists a mapping $p : V \to \mathbb{R}^2$ from its vertices to the plane such that $uv \in E$ if and only if the Euclidean distance between $p(u)$ and $p(v)$ is at most 1. Any explicit mapping p that certifies that G is a unit disk graph is called an *embedding*. When we have an embedding p, we often make no distinction between a vertex u and its corresponding point $p(u)$ in the plane.

The class of unit disk graphs is popular due to its elegance and its versatility in capturing real-world optimisation problems [5]. For example, an embedded unit disk graph may represent placement of transceivers so that circles of radius 1/2 centred at the points represent transmission areas. Indeed, the class was originally introduced in 1980 to model frequency assignment [10], with chromatic number one of the first studied parameters. Clark, Colbourn and Johnson [5] published a proof that it is NP-hard to compute the chromatic number of unit disk graphs. They also showed the clique number of unit disk graphs is polynomial-time computable. Therefore, any upper bound C on the extremal ratio

$$r := \sup\{\chi(G)/\omega(G) \,|\, G \text{ is a unit disk graph}\}$$

(algorithmic or not) implies an efficient C-approximation of the chromatic number: simply output $C\cdot\omega(G)$. In 1991, Peeters [19] noted a simple 3-approximation: after lexicographically ordering the vertices of G according to any fixed embedding, a basic geometric argument proves that G is $3(\omega(G)-1)$-degenerate (and then apply (1)). Since 3-colourability of unit disk graphs is NP-complete, there is no efficient $(4/3-\varepsilon)$-approximation unless P=NP. Moreover, Malesińska, Piskorz and Weißenfels [16] exhibited some unit circular-arc graphs that certify $r \geq 3/2$. To date, the best approximation ratio known is 3.

Mahdian [14, 15] showed in 2000 that it is NP-hard to compute the strong chromatic index, even restricted to bipartite graphs of large fixed girth. More recently, Chalermsook, Laekhanukit and Nanongkai [4] showed that in general there is no polynomial-time $(n^{1/3-\varepsilon})$-approximation algorithm (where n is the number of vertices in the input) unless NP=ZPP.

To the best of our knowledge, no previous work has shown NP-hardness upon restriction to the class of unit disk graphs. Nevertheless, Barrett *et al.* [1] have initiated the study of approximate strong edge-colouring for unit disk graphs. With an argument similar to Peeters' [19] for chromatic number, they showed that $\delta^*(L(G)^2) \leq 8\omega_2'(G)$ for any unit disk graph G, which by (2) certifies that the greedy algorithm is an efficient 8-approximation for the strong chromatic index. Kanj, Wiese and Zhang [13] noted an efficient online 10-approximation for the strong chromatic index with essentially the same analysis as in [1].

3 Main results

Our work improves on [1] in several ways.

▶ **Theorem 3.1.** *For any unit disk graph G, $\delta^*(L(G)^2) \leq 6(\omega_2'(G)-1)$.*

▶ **Corollary 3.2.** *The greedy algorithm under a reverse degeneracy ordering of the edges is an efficient 6-approximation for the strong chromatic index of unit disk graphs.*

The proof of Theorem 3.1 is rather involved. It shows that, for any embedded unit disk graph, some well-chosen edge-ordering certifies the required degeneracy bound. It would be very interesting to improve on the approximation ratio of 6.

▶ **Theorem 3.3.** *For any unit disk graph G, $\Delta(L(G)^2) \leq 8(\chi_2'(G)-1)$.*

▶ **Corollary 3.4.** *The greedy algorithm is an efficient online 8-approximation[1] for the strong chromatic index of unit disk graphs.*

To prove Theorem 3.3, it suffices to solve the following kissing number-type problem. Given two intersecting unit disks C_1 and C_2 in $\mathbb{R}^2$, what is the size of a largest collection of pairwise non-intersecting unit disks such that each one intersects $C_1 \cup C_2$? The corresponding problem in $\mathbb{R}^3$ seems quite natural.

[1] To avoid any ambiguity, in the online setting *vertices* are revealed one at a time and all edges between a newly revealed vertex and previous vertices must be immediately and irrevocably assigned a colour.

▶ **Theorem 3.5.** *Strong edge-k-colourability of unit disk graphs is NP-complete, where $k = 6$ or $k = \binom{\ell}{2} + 4\ell + 6$ for some fixed $\ell \geq 5$.*

▶ **Corollary 3.6.** *It is NP-hard to compute the strong chromatic index of unit disk graphs. Moreover, it cannot be efficiently $(7/6 - \varepsilon)$-approximated unless P=NP.*

For $k \leq 3$, strong edge-k-colouring is polynomially solvable. It remains open to determine the complexity when k is in $\{4, 5\}$. The proof of Theorem 3.5 borrows upon ideas in the work of Gräf, Stumpf and Weißenfels [9], but with extra non-trivial difficulties for strong edge-colouring.

4 Further discussion

We can state slightly more general versions of our approximation results that give a more geometric flavour. We call a graph $G = (V, E)$ a *twin unit disk graph* if there exists a mapping $p : V \to \mathbb{R}^2 \times \mathbb{R}^2$ from its vertices to pairs of points in the plane such that

- the Euclidean distance between $p(u)_1$ and $p(u)_2$ is at most 1 for every $u \in V$; and
- $uv \in E$ if and only if the Euclidean distance between $p(u)_1$ and $p(v)_1$, between $p(u)_1$ and $p(v)_2$, between $p(u)_2$ and $p(v)_1$, or between $p(u)_2$ and $p(v)_2$ is at most 1.

Equivalently, this is the intersection class over unions of pairs of intersecting unit disks in $\mathbb{R}^2$. Note that, for any unit disk graph G, both G and $L(G)^2$ are twin unit disk graphs. Indeed for any edge $e = (p_1, p_2)$ in a unit disk graph, e can be represented by a vertex u in a twin unit disk with $p(u)_1 = p_1$ and $p(u)_2 = p_2$. So it is NP-hard to determine the chromatic number of twin unit disk graphs. We actually found the following stronger versions of Theorems 3.1 and 3.3, which imply efficient 6-approximation and online 8-approximation for the chromatic number of twin unit disk graphs (by (1)).

▶ **Theorem 4.1.** *For any twin unit disk graph G, $\delta^*(G) \leq 6(\omega(G) - 1)$.*

▶ **Theorem 4.2.** *For any twin unit disk graph G, $\Delta(G) \leq 8(\chi(G) - 1)$.*

If we were able to efficiently compute or well approximate the clique number of twin unit disk graphs or, in particular, the strong clique number of unit disk graphs, then we would have a strong incentive to bound $r_2' := \sup\{\chi_2'(G)/\omega_2'(G) \mid G \text{ is a unit disk graph}\}$. This is a natural optimisation problem regardless. We only know $r_2' \leq 6$ by Theorem 3.1, and $r_2' \geq 4/3$ by considering the cycle C_7 on seven vertices (since $\chi_2'(C_7) = 4$ while $\omega_2'(C_7) = 3$). Relatedly, we believe that the following problem is worth investigating.

▶ Conjecture 4.3. It is NP-hard to compute the clique number of twin unit disk graphs.

5 A short proof for a 7-approximation

For expository purposes, we present a much shorter argument for a weaker approximation than Theorem 4.1. The proof is nearly the same as what Barrett *et al.* [1] used for an upper bound on the approximation ratio of 8, but with a small twist.

▶ Proposition 5.1. For any twin unit disk graph G, $\delta^*(G) \leq 7(\omega(G) - 1)$.

Proof. Let $G = (V, E)$ be a twin unit disk graph. Fix any embedding $p : V \to \mathbb{R}^2 \times \mathbb{R}^2$ of G in the plane. Equipped with such an embedding, we first define an ordering of V and then use it to certify the promised degeneracy property.

The ordering we use for this result, a lexicographic ordering, is the same used in [1]. Let $(x_1, y_1), (x_2, y_2), \ldots$ be a sequence of points in $\mathbb{R}^2$ defined by listing the elements of

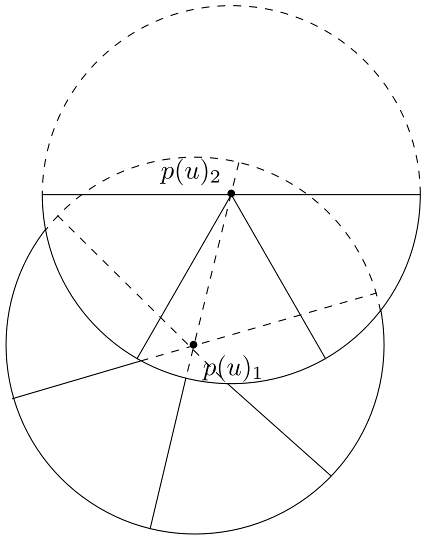

Figure 1 The seven sectors one of which must contain $p(v)_1$ or $p(v)_2$.

$\bigcup_{u\in V}\{p(u)_1, p(u)_2\}$ according to the lexicographic order on $\mathbb{R}^2$ (i.e. (a, b) is before (c, d) if and only if $a < c$ or ($a = c$ and $b \le d$)). We consider the points of this sequence in order and add vertices at the end of our current ordering of V as follows. When considering point (x_j, y_j) for some $j \ge 1$, we add all vertices $u \in V$ for which there is some $i \le j$ such that $\{p(u)_1, p(u)_2\} = \{(x_i, y_i), (x_j, y_j)\}$, and we do so according to the lexicographic order on $\mathbb{R}^2$.

It suffices to show that each vertex $u \in V$ has at most $7(\omega(G) - 1)$ neighbours that precede it in the lexicographic ordering. To do so, we show that every such neighbour v of u satisfies that either $p(v)_1$ or $p(v)_2$ is contained in one of seven unit $(\pi/3)$-sectors (each of which is centred around either $p(u)_1$ or $p(u)_2$). This is enough, since the set of vertices that map one of their twin points into one such sector induces a clique in G that includes u.

Let $u \in V$ and suppose without loss of generality that $p(u)_1$ is before $p(u)_2$ in lexicographic order. First observe that, if $v \in V$ is before u in the lexicographic order, then both $p(v)_1$ and $p(v)_2$ must be in the region of $\mathbb{R}^2$ that has smaller or equal y-coordinate compared to $p(u)_2$. If, moreover $uv \in E$, then $p(v)_1$ or $p(v)_2$ must lie in either a unit half-disk centred at $p(u)_2$ or in the unit disk centred at $p(u)_1$. We partition the unit disk centred at $p(u)_1$ into six unit $(\pi/3)$-sectors such that the line segment $[p(u)_1, p(u)_2]$ lies along the boundary between two of the sectors. Note that any of the points in the two sectors incident to $[p(u)_1, p(u)_2]$ also lies in the unit disk centred at $p(u)_2$. See Figure 1. Therefore, the four other sectors together with the three sectors that partition the unit half-disk centred at $p(u)_2$ are the seven unit $(\pi/3)$-sectors that we desire. ◀

It turns out that for Theorem 4.1 we can take the same approach as in Proposition 5.1, except with an ordering that is more subtle and an analysis that is substantially longer and more difficult. Full proof details will be made available in a journal version of the manuscript (also on arXiv).

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