Recognizing Visibility Graphs of Polygons with Holes

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Abstract

The visibility graph of a polygon corresponds to its internal diagonals and boundary edges. For each vertex on the boundary of the polygon, we have a vertex in this graph and if two vertices of the polygon see each other there is an edge between their corresponding vertices in the graph. Two vertices of a polygon see each other if and only if their connecting line segment completely lies inside the polygon, and they are externally visible if and only if this line segment completely lies outside the polygon. Recognizing visibility graphs is the problem of deciding whether there is a simple polygon whose visibility graph is isomorphic to a given input graph. This problem is well-known and well-studied, but yet widely open in geometric graphs and computational geometry.

Existential Theory of the Reals is the complexity class of problems that can be reduced to the problem of deciding whether there exists a solution to a quantifier-free formula $F(X_1, X_2, ..., X_n)$, involving equalities and inequalities of real polynomials with real variables. The complete problems for this complexity class are called $\exists\mathbb{R}$-*Complete*.

In this paper, we show that recognizing visibility graphs of polygons with holes is $\exists\mathbb{R}$-*Complete*.

1 Introduction

The visibility graph of a simple planar polygon is a graph in which there is a vertex for each vertex of the polygon and for each pair of visible vertices of the polygon there is an edge between their corresponding vertices in this graph. Two points in a simple polygon are visible from each other if and only if their connecting segment completely lies inside the polygon. In this definition, each pair of adjacent vertices on the boundary of the polygon are assumed to be visible from each other. This implies that we always have a Hamiltonian cycle in a visibility graph which determines the order of vertices on the boundary of the corresponding polygon. A polygon with holes has some non-intersecting holes inside the boundary of the polygon. In these polygons the area inside a hole is considered as the outside area and internal and external visibility graphs of such polygons are defined in the same way as defined for simple polygons. In the visibility graph of a polygon with holes, we have the sequence of vertices corresponding to the boundary of each hole, as well.

Computing the visibility graph of a given simple polygon has many applications in computer graphics [19], computational geometry [11] and robotics [2]. There are several efficient polynomial time algorithms for this problem [11].

This concept has been studied in reverse as well: Is there any simple polygon whose visibility graph is isomorphic to a given graph, and, if there is such a polygon, is there any way to reconstruct it(finding positions for its vertices in the plane)? The former problem is

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known as recognizing visibility graphs and the latter one is known as reconstructing a polygon from a visibility graph. The computational complexity of these problems are widely open. The only known result about the computational complexity of these problems are that they belong to *PSPACE* [7] complexity class. More precisely, they belong to the class of *Existential theory of the reals* [15]. This means that it is not even known whether these problems are in *NP* or can be solved in polynomial time. Even, if we are given the Hamiltonian cycle of the visibility graph which determines the order of vertices on the boundary of the target polygon, the exact complexity classes of these problems are still unknown.

However, these problems have been solved efficiently for special cases of *tower* and *spiral polygons*. The recognizing and reconstruction problems have been solved for tower polygons [6] and spiral polygons [8] in linear time in terms of the size of the graph.

Although there is some progress on recognizing and reconstruction problems, there have been plenty of studies on characterizing visibility graphs. In 1988, Ghosh introduced three necessary conditions for visibility graphs and conjectured their sufficiency [9]. In 1990, Everett proposed a graph that rejects Ghosh's conjecture [7]. She also refined Ghosh's third necessary condition to a new stronger one [10]. In 1992, Abello *et al.* built a graph satisfying Ghosh's conditions and the stronger version of the third condition which was not the visibility graph of any simple polygon [1], disproving the sufficiency of these conditions. In 1997, Ghosh added his forth necessary condition and conjectured that this condition along with his first two conditions and the stronger version of the third condition are sufficient for a graph to be a visibility graph. Finally, in 2005 Streinu proposed a counter example for this conjecture [18].

Existential theory of the reals ($\exists\mathbb{R}$) is a complexity class that was implicitly introduced in 1989 [3], introduced by Shor in 1991 [17] and explicitly defined by Schaefer in 2009[16]. It is the complexity class of problems which can be reduced to the problem of deciding, whether there is a solution for a Boolean formula $\phi : \{True, False\}^n \to \{True, False\}$ in propositional logic, in the form $\phi(F_1(X_1, X_2, ..., X_N), F_2(X_1, X_2, ..., X_N), ..., F_n(X_1, X_2, ..., X_N))$, where each $F_i : \mathbb{R}^N \to \{True, False\}$ consists of a polynomial function $G_i : \mathbb{R}^N \to \mathbb{R}$ on some real variables, compared to 0 with one of the comparison operators in $\{<, \leq, =, >, \geq\}$ (for example $G_i(X_1, X_2) = X_1^3 X_2^2 - X_1 X_2^3$ and $F_i(X_1, X_2) \equiv G_i(X_1, X_2) < 0$). Clearly, satisfiability of quantifier free Boolean formulas belong to $\exists\mathbb{R}$. Therefore, $\exists\mathbb{R}$ includes all NP problems. In addition, $\exists\mathbb{R}$ belongs to $PSPACE$ [5] and we have $NP \subseteq \exists\mathbb{R} \subseteq PSPACE$. Many other decision problems, especially geometric problems, belong to $\exists\mathbb{R}$ and some are complete for this complexity class. Recognizing *LineArrangement* (*Stretchability*), simple order type, intersection graphs of segments, recognizing visibility graphs of a point set, and intersection graphs of unit disks in the plane are some problems which are complete for $\exists\mathbb{R}$ or simply $\exists\mathbb{R}$-*Complete* [5]. The computational complexity of these problems was open for years and after proving $\exists\mathbb{R}$-*Completeness*, the study of the $\exists\mathbb{R}$ class and $\exists\mathbb{R}$-*Complete* problems gets more attention in computational geometry literature. We discuss the problem, Recognizing *LineArrangement* (*Stretchability*), in more details in this paper in Section 2.

In this paper, we show that recognizing a visibility graph of polygon with holes is $\exists\mathbb{R}$-*Complete*. In this problem we assume that the sequence of vertices corresponding to the boundary of the polygon and its holes, is given as input[1]

[1] While (in Dec-2017) we submitted this result to SOCG2018 and later submitted it to arXiv in Apr-2018[4], in an independent work by Hoffmann and Merckx[13] in Jan-2018 they used another technique to prove the $\exists\mathbb{R}$-*Completeness* of recognizing the visibility graphs of polygon with holes. First, they proved the $\exists\mathbb{R}$-*Completeness* of recognizing the *AllowableSequences* and then reduced this problem to recognizing the visibility graphs of polygon with holes.

2 Preliminaries and Definitions

2.1 Line arrangement and stretchability

Considering a set of lines in the plane, the problem of describing their arrangement is called *LineArrangement*. This is an important and fundamental problem in combinatorics and a well-studied problem in computational geometry. This description for a set of lines $l_1, l_2, ..., l_n$ consists of their vertical order with respect to a vertical line to the left of all their intersections, and for each line l_i, the order of lines that are intersected by l_i when we traverse l_i from left to right (we assume that none of the input lines l_i is vertical). Recognizing whether there can be a set of lines in the plane with the given *LineArrangement*, is called *Recognizing LineArrangement* or simply *LineArrangement* problem. When the lines are in general position (all pairs of lines intersect and no 3 lines intersect at the same point) the problem is called *SimpleLineArrangement*. It has been proved that *SimpleLineArrangement* is $\exists\mathbb{R}$-*Complete* [5, 14].

A pseudo-line is a monotone curve with respect to the X axis. Assuming that no pair of pseudo-lines intersect each other more than once, we can describe an instance of recognizing *PseudoLineArrangement* problem in the same way as we did for *LineArrangement*. However, Recognizing *PseudoLineArrangement* belongs to the P complexity class and it can be decided with a Turing machine in polynomial time [12]. A pseudo code implementation and the details of this algorithm has been given in [4] and depicted in Figure 1.

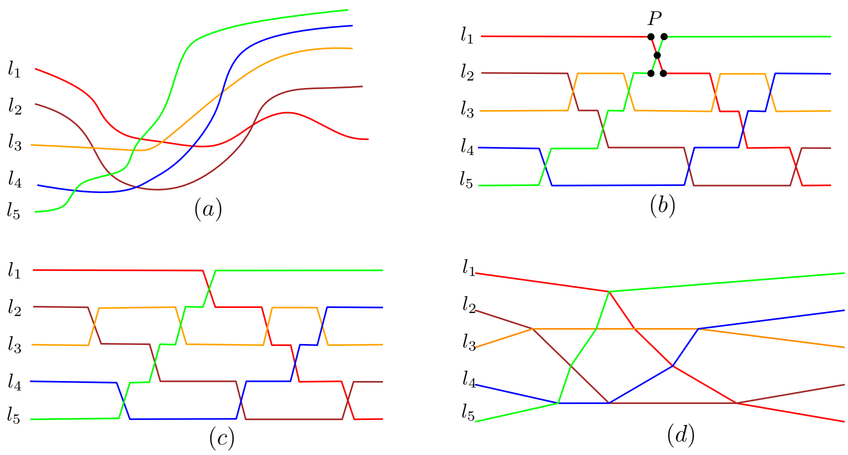

Figure 1 The reconstruction algorithm for *PseodoLineArrangement*.

Trivially, if an instance of the *LineArrangement* problem is realizable, it has a *PseudoLineArrangement* realization as well. On the other hand, if an instance of the *PseudoLineArrangement* problem has a realization in which all segments of each pseudo-line lie on the same line, the input instance has also a *LineArrangement* realization as well.

Therefore, we can describe the *LineArrangement* problem as follows:

- Is it possible to stretch a *PseudoLineArrangement* of a given line arragement description such that each pseudo-line lies on a single line?

This problem is known as *Stretchability*. As stated before, pseudo-line arragement belongs to the P complexity class and can be recognized and reconstructed efficiently. Therefore, $\exists\mathbb{R}$-*Completeness* of *LineArrangement* implies that *Stretchability* is $\exists\mathbb{R}$-*Complete*.

2.2 Visibility graph of a polygon with holes

A polygon with holes is a simple polygon that has a set of non-colliding areas (simple polygons) inside it. The internal areas of the holes belong to the outside area of the polygon. In these polygons, two vertices are visible from each other if their connecting segment completely lies inside the polygon. The visibility graph of a polygon with holes is a graph whose vertices correspond to the vertices of the polygon and the holes, and in this graph there is an edge between two vertices if and only if their corresponding vertices in the polygon are visible from each other (see Fig. 2). In this paper, we assume that along with the visibility graph, we have the cycles that correspond to the order of vertices on the boundary of the polygon and the holes. The cycle that corresponds to the external boundary of the polygon is called the external cycle(see Fig. 2).

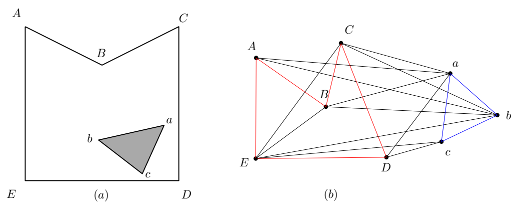

Figure 2 A polygon with one hole (a), and its visibility graph (b).

3 Complexity of Recognizing Visibility Graphs of Polygons with Holes

In this section, we show that recognizing a visibility graph of polygon with holes is $\exists\mathbb{R}$-*Complete*. This is done by reducing an instance of the stretchability problem to an instance of this problem.

In Section 2.1 we showed that we can describe the line arragement problem as an instance of stretchability of pseudo-lines in which each pseudo-line is composed of a chain of segments and the break-points of these chains(except the first and the last endpoints of the chains) correspond to the intersection points of the pseudo-lines. We build a visibility graph $\mathcal{G}$, an external cycle $\mathcal{P}$, and a set of boundary cycles $\mathcal{H}$ from an instance of such a stretchability problem, and prove that the pseudo-line arragement is stretchable in the plane if and only if there exists a polygon with holes whose visibility graph is $\mathcal{G}$, its external cycle is $\mathcal{P}$ and the set of boundary cycles of its holes is $\mathcal{H}$.

Assume that $(\mathcal{L}, \mathcal{S})$ is an instance of the stretchability problem where, as described in [4], $\mathcal{L} = \langle l_1, l_2, ..., l_n \rangle$ is the sequence of the pseudo-lines and $\mathcal{S} = \langle S_1, S_2, ..., S_n \rangle$ is the sequence of the intersections of these pseudo-lines in which $S_i = \langle l_{a(i,1),...,a(i,n-1)} \rangle$ is the order of lines intersected by l_i. Let denote by $(\mathcal{G}, \mathcal{P}, \mathcal{H})$ the corresponding instance of the visibility graph realization in which $\mathcal{G}$ is the visibility graph, $\mathcal{P}$ is the external cycle of the outer boundary of the polygon and $\mathcal{H} = \{H_1, H_2, ..., H_k\}$ is the set of boundary cycles of its holes. To build this instance, consider an example of such an $(\mathcal{L}, \mathcal{S})$ instance shown in Fig. 3-a. This figure shows a pseudo-line realization obtained from the pseudo-line reconstruction algorithm for an instance of four pseudo-lines. If this instance is stretchable, like the one shown in Fig. 3-b, we can build a polygon with holes like the one shown in Fig. 3-c. The outer boundary of this polygon and the boundary of its holes lie along a set of convex curves connecting the endpoints of each stretched pseudo-line. Precisely, for each stretched pseudo-line l_i, as in Fig. 3-b, there is a pair of convex chains on both of its sides which connect its endpoints. These pair of convex chains are sufficiently close to their corresponding stretched pseudo-lines, and their break-points are the intersection points of these chains(like point o in Fig. 3-c). This pair of convex chains, for each pseudo-line l_i, makes a convex polygon which is called its channel and is denoted by $Ch(l_i)$. The outer boundary of the target polygon and the boundary of its holes are obtained by removing those segments of the chains that lie inside another channel (see Fig. 3-c). Note that, we do not have the stretched realization of $(\mathcal{L}, \mathcal{S})$ instance of the stretachability problem. But, from the pseudo-line realization, we can determine $\mathcal{G}$, $\mathcal{P}$ and $\mathcal{H}$ of the corresponding instance $(\mathcal{G}, \mathcal{P}, \mathcal{H})$ in polynomial time. As shown in Fig. 3-d, $\mathcal{P}$ and $\mathcal{H}$ are obtained by imaginary drawing a channel for each pseudo-line l_i. Finally, the vertex set of graph $\mathcal{G}$ is the set of all break-points of these convex chains, and, two vertices are connected by an edge if and only if they belong to the boundary of the same channel. The following theorem shows the relationship between $(\mathcal{L}, \mathcal{S})$ and $(\mathcal{G}, \mathcal{P}, \mathcal{H})$ problem instances. The detailed proof of the theorem is given in [4].

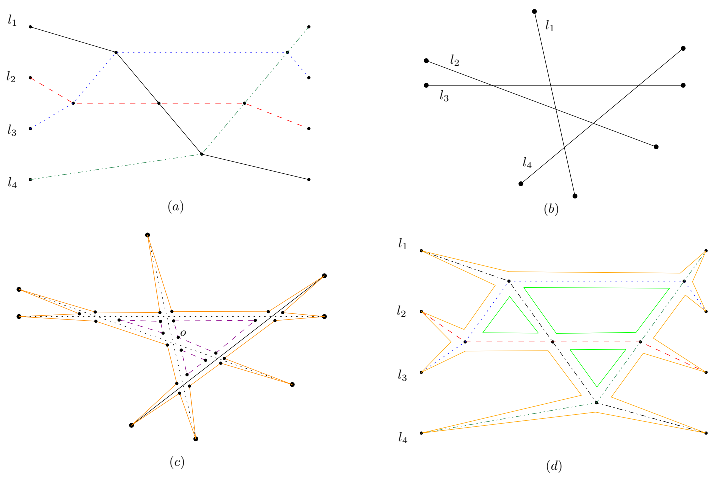

Figure 3 A polygon with holes which is constructed from an instance of the *PseudoLineArrangement* problem.

▶ **Lemma 1.** *An instance $(\mathcal{L}, \mathcal{S})$ of the stretchability problem is realizable if and only if its corresponding $(\mathcal{G}, \mathcal{P}, \mathcal{H})$ instance of the visibility graph is realizable.*

It is easy to show that recognizing a visibility graph of a polygon with holes belongs to $\exists\mathbb{R}$. It can be done by constructing a set of boolean formulas on a set of functions $F_i : \mathbb{R} \to \mathbb{R}$ on the set of vertices (a pair of two real numbers) of the polygon with hole, that verifies the visibility constrains in it. While the stretchability problem is $\exists\mathbb{R}$-*Complete* and our reduction is polynomial, Theorem 1 implies the $\exists\mathbb{R}$-*Hardness* of recognizing visibility graph of polygon with holes. Therefore, we have the following theorem.

▶ **Theorem 2.** *Recognizing visibility graph of polygon with holes is $\exists\mathbb{R}$-Complete.*

4 Conclusion

In this paper, we showed that the visibility graph recognition problem is $\exists\mathbb{R}$-*Complete* for polygons with holes.

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