On Plane Subgraphs of Complete Topological Drawings

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Abstract

We consider plane subgraphs of simple topological drawings of K_n, and in particular maximal ones. Fulek and Ruiz-Vargas showed that between any plane connected subgraph F and a vertex v not in F, there are two edges from v to F not crossing F. We give an $O(n)$ time algorithm to find such edges, and show that the result also holds if F is disconnected. In particular, any plane subgraph can be augmented to a 2-connected one. This leads to our main structural result, showing that maximal plane subgraphs are 2-connected and what we call *essentially 3-edge-connected*.

1 Introduction

In a *topological drawing* (in the plane or on the sphere) of a graph, vertices are represented by points and edges are arcs with its two vertices as endpoints. It is *simple* if two edges intersect at most in a single point, either at a common endpoint or at a crossing in their relative interior. Let D_n be a simple topological drawing of the complete graph K_n on n vertices. Clearly, all straight-line drawings are simple topological drawings, and thus problems on embedding graphs on a set of points usually generalize to finding subgraphs of D_n. Such problems are often concerned with crossing-free (i.e., *plane*) subgraphs. (Herein, we consider graphs in connection with their drawings, and in particular when addressing subgraphs of K_n we also consider the associated sub-drawing of D_n.) Crossing-free edge sets in D_n have attracted considerable attention. Pach, Solymosi, and Tóth [4] showed that any D_n has $\Omega\left(\log^{1/6}(n)\right)$ pairwise disjoint edges. This bound was subsequently improved [5, 1, 8]. The current best bound of $\Omega(n^{1/2-\epsilon})$ is by Ruiz-Vargas [7]. In the course of their work on disjoint edges and empty triangles in D_n, Fulek and Ruiz-Vargas [2] showed the following lemma.

▶ **Lemma 1.1** (Fulek and Ruiz-Vargas [2]). *Between any plane connected subgraph F of D_n and a vertex v not in F, there exist at least two edges from v to F that do not cross F.*

In this work, we show that such edges incident to v can be found in $O(n)$ time. Further, we extend their result to disconnected plane subgraphs. It turns out that any plane subgraph of D_n can be augmented to a 2-connected plane subraph of D_n. Maximal plane subgraphs of D_n have further interesting properties. For example, we show that, when removing two edges, they either stay connected or one of the two components is a single vertex.

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This is an extended abstract of a presentation given at EuroCG'19. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

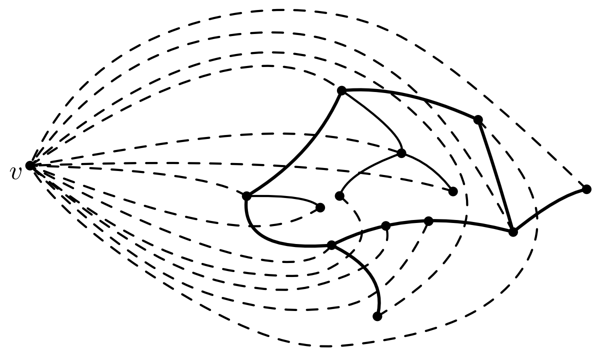

Figure 1 The order of the first intersections of $S(v)$ along the face matches the rotation of v.

2 Adding a single vertex

Lemma 1.1 turns out to be a useful workhorse for identifying plane subgraphs. We provide an efficient algorithm for computing the uncrossed edges. We assume that a topological drawing D_n of K_n with vertex set $V = \{v_1, \dots, v_n\}$ is given by its rotation system and the inverse rotation system: The *rotation* of a vertex $v_i \in V$ is a permutation of $V \setminus \{v_i\}$ given by the circular order in which the edges to all other vertices emanate from v_i. We denote these edges by $S(v)$, i.e., the *star* with center v. The *inverse rotation system* that, for each vertex v_i and each index $j \neq i$ gives the index of v_j in the (linearized) rotation of v_i. (It can be obtained from the rotation system in $O(n^2)$ time.) Using these two structures, it is well-known that one can determine whether two edges cross, in which direction an edge is crossed, and in which order two non-crossing edges cross a third one in constant time [3].

▶ **Theorem 2.1.** *Given a simple topological drawing of K_n, a connected plane subgraph F, and a vertex v not in F, we can find the edges from v to F not crossing F in $O(n)$ time.*

Proof. W.l.o.g., let the face that contains v be the outer face f. For each edge from v to F, consider its first intersection with F. The order in which these points are encountered in a clockwise walk of the boundary of f matches the rotation of v (restricted to F); see Figure 1. (Edges walked twice can be seen as two “half-edges”, essentially treating f as a cycle.) The algorithm starts by finding, for any edge vw_1, the edge of F that intersects vw_1 closest to v along vw_1. Using the rotation system and its inverse, this edge e_1 of face f can be found in linear time. We keep uncrossed edges of $S(v)$ on a stack σ and walk the boundary of f and the rotation of v, in each step making progress in at least one of them or removing an edge from σ. Let vw_i be the next edge in the counterclockwise rotation of v (initially vw_2). If vw_i crosses e_k (initially e_1), we iterate considering the counterclockwise successor vw_{i+1} of vw_i in the rotation of v. If there is no intersection of vw_i and e_k, we iterate with the clockwise successor e_{k+1} of e_k in f instead of e_k, after popping all edges from σ that cross e_{k+1}. We do the same if w_i is the clockwise endpoint of e_k along f; in addition, if vw_i is clockwise between e_k and e_{k+1} in the rotation of w_i, we put vw_i on σ and in the next iteration consider vw_{i+1}. Note that in a generic step σ contains the explored edges of $S(v)$ uncrossed by the explored edges $e_1, \dots, e_{k-1}$ of f. Eventually, σ contains the uncrossed edges of $S(v)$. ◀

We now discuss an extension of Lemma 1.1 to plane subgraphs. This will also follow independently from Theorem 3.1. Still, our result (proven in the full version) gives further insight on the position of the edges in the rotation of the additional vertex v.

Let F be a plane subgraph of D_n. The edges of star $S(v)$ are called *rays*. Suppose that ray vr first crosses the edge pq of F. W.l.o.g., we suppose that the rays vr, vp, and vq appear in this clockwise order around v. Let x be the crossing of ray vr and edge pq. We define the clockwise range R_{cw} of rays centered at v corresponding to crossing x: if no ray in the range (vp, vq) crosses edge pq between x and p, then R_{cw} is the set of rays in the clockwise range $(vr, vp]$ (i.e., including vp but not vr); otherwise, if some rays in the clockwise range (vp, vq) cross edge pq between x and p, then R_{cw} is the set of rays in clockwise range (vr, vl), where vl is the ray in the range (vp, vq) crossing edge e between x and p in a point y closest to x along pq. See Figure 2, which also indicates the analogous definition of the counterclockwise range R_{ccw}.

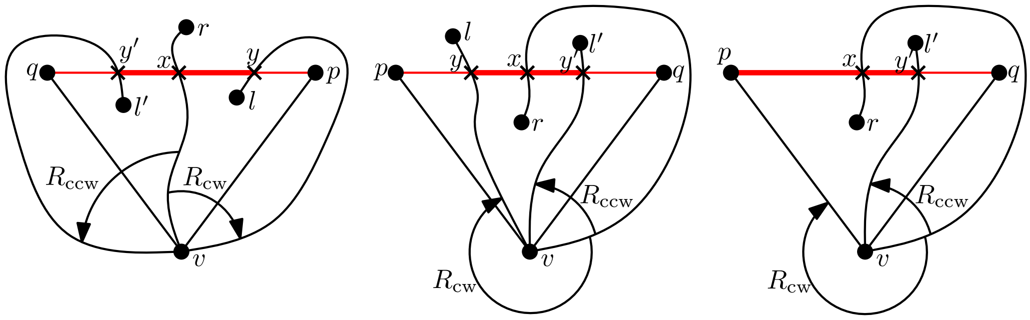

Figure 2 The clockwise and counterclockwise ranges of a first crossing.

▶ **Proposition 2.2.** *Suppose ray vr first crosses edge e of F at point x. Let R_{cw} and R_{ccw} be the ranges of rays of v corresponding to that crossing. Then, each one of those two ranges contains an uncrossed ray. As a consequence, $S(v)$ contains at least two uncrossed rays.*

3 Structure of maximal plane subgraphs

▶ **Theorem 3.1.** *A maximal plane subgraph of any D_n is spanning and 2-connected.*

Proof. The proof is by induction on n. The result is obviously true for $n \leq 3$. For $n > 3$, suppose there exists a maximal plane subgraph $\bar{F}$ that is not 2-connected.

We first argue that, under this assumption, $\bar{F}$ does not have a vertex v of degree less than 3. Suppose the contrary and let F' be the subgraph of $\bar{F}$ after removing v, and let $\bar{F}'$ be a maximal plane subgraph (in the drawing $D_n - \{v\}$ of K_{n-1}) containing F'; by the induction hypothesis, $\bar{F}'$ is 2-connected. We observe that v cannot have degree less than 2, as applying Lemma 1.1 to v and $\bar{F}'$ would give two edges at v not crossing $\bar{F}$, contradicting the maximality of $\bar{F}$. Suppose v has degree 2. As we assume that $\bar{F}$ is not 2-connected, F' cannot be 2-connected. However, $\bar{F}'$ is 2-connected, and hence there exists an edge e' in $\bar{F}' - F'$. By the maximality of $\bar{F}$, e' must cross at least one edge vw of $\bar{F}$ incident to v. But applying Lemma 1.1 to v and $\bar{F}'$ gives at least two edges incident to v not crossing $\bar{F}'$. These two edges and also vw do not cross $\bar{F}$, contradicting the maximality of $\bar{F}$.

Assume that $\bar{F}$ is not connected. Let C_1, C_2 be two connected components of $\bar{F}$. As all vertices have degree at least 3, C_1 cannot be an outerplanar graph, and thus has more than one face. W.l.o.g., the unbounded face contains C_2. Let v_1 be an interior vertex of C_1. Let F' be the graph obtained from $\bar{F}$ by removing v_1, and let f_1 be the face in F' that contains v_1. The face containing C_2 remains unchanged by the removal. By induction, F'

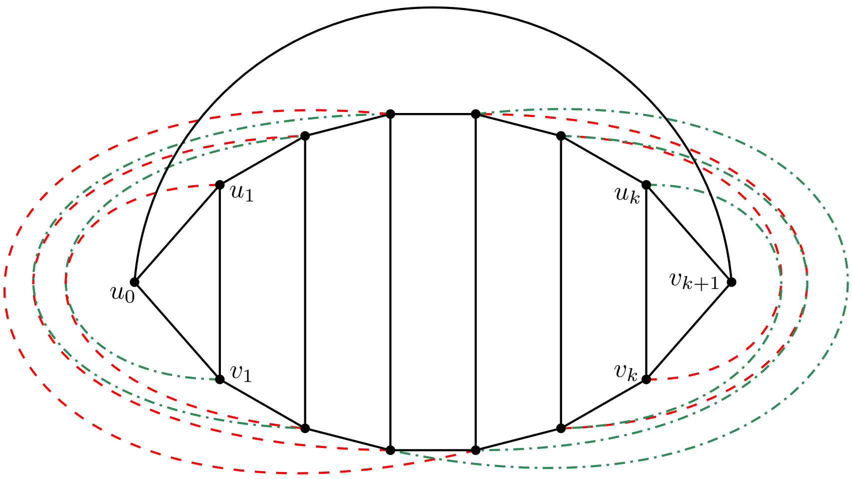

Figure 3 The black edges form a maximal plane subgraph with $\lceil 3n/2 \rceil$ edges. The missing edges should be drawn as straight line segments inside the convex hull of the set of points.

can be completed to a 2-connected plane graph $\bar{F}'$. Due to the maximality of $\bar{F}$, all edges in $\bar{F}' - F'$ must be in f_1. As C_2 is outside f_1, $\bar{F}'$ could not be connected. Thus, $\bar{F}$ is connected.

By a similar reasoning we arrive at our contradiction to $\bar{F}$ not being 2-connected. A *block* is a 2-connected component of a graph, and a *leaf block* is a block with only one cut vertex. As $\bar{F}$ is not 2-connected, it has at least two leaf blocks B_1 and B_2. As all vertices have degree at least 3, B_1 cannot have all its vertices on the same face. Again, w.l.o.g., B_2 is in the outer face of B_1, and there is an interior vertex v_1 of B_1. Removing v_1 from $\bar{F}$, we obtain a plane graph F' that has a face f_1 containing v_1, and F' is contained in a maximal plane graph $\bar{F}'$ that is 2-connected. Again, by maximality of $\bar{F}$, all edges in $\bar{F}' - F'$ must be in f_1. However, this contradicts the fact that $\bar{F}'$ is 2-connected. Hence, $\bar{F}$ must be 2-connected. ◀

Theorem 3.1 gives a means of obtaining more properties of maximal plane subgraphs.

▶ **Lemma 3.2.** *If a maximal plane subgraph $\bar{F}$ of D_n contains a vertex v of degree 2, then the subgraph of $\bar{F}$ obtained after removing v is also maximal in $D_n - \{v\}$.*

▶ **Proposition 3.3.** *Any maximal plane subgraph $\bar{F}$ of D_n with $n \geq 3$ must contain at least* $\min(\lceil 3n/2 \rceil, 2n-3)$ *edges. This bound is tight.*

A sketch for showing tightness of $\lceil 3n/2 \rceil$ edges is given in Figure 3.

▶ **Lemma 3.4.** *Let $C = (v_1, v_2, \ldots, v_k)$ be a plane cycle of D_n, $k \geq 3$, with faces f_1 and f_2. If there is no diagonal of C entirely in f_1, then all diagonals of C are entirely in f_2.*

Proof. The proof is by induction on k. For $k < 5$ the statement is obvious, so suppose $k \geq 5$ and consider only the subgraph induced by the vertices of C. By Lemma 3.2, there must exist a diagonal placed in f_2 connecting two vertices at distance 2 on C. W.l.o.g., let this diagonal be $v_k v_2$ and let Δ be the triangle $v_k v_1 v_2$. Then the cycle $C_1 = (v_2, v_3, \ldots, v_k)$ with $k-1$ vertices has the faces $f_1' = f_1 + \Delta$ and $f_2' = f_2 - \Delta$.

We argue that there cannot be diagonals of C_1 entirely in f_1'. Such a diagonal e would have to intersect Δ. When adding e to $C \cup \{v_k v_2\}$ and removing all edges crossed by e, we obtain a plane graph F in which v_1 has degree 0 or 1. By Lemma 1.1, there must be another edge between v_1 and C_1, and this would be a diagonal of C entirely in f_1. Thus, by induction, any diagonal v_iv_j of C_1 is entirely in f_2' and hence also in f_2. It remains to see that the diagonals with endpoint v_1 are also in f_2. By our induction hypothesis, diagonal v_2v_4 is in f_2' and thus also in f_2. Hence, applying the hypothesis on the cycle $C_3 = (v_1, v_2, v_4, \ldots, v_k)$ we deduce that all the diagonals incident to v_3 must be in f_2. So it remains to see that diagonal v_1v_3 is also in f_2. But also v_3v_5 is in f_2', so it is also in f_2, and again by induction on the cycle $(v_1, v_2, v_3, v_5, \ldots, v_k)$, all the diagonals not incident to v_4 are also in f_2. ◀

It was previously known that even for the case where there are diagonals intersecting both faces, there are at least $\lceil k/3 \rceil$ of them not crossing C (cf. [6, Corollary 6.6]); Proposition 3.3 implies that for $k \geq 6$, there are at least$\lceil k/2 \rceil$ diagonals not crossing C.

To prove the next result we recall some definitions and properties of any 2-connected graph $G = (V, E)$. Two vertices v_1, v_2 are called a *separation pair* of G if the induced subgraph $G \setminus \{v_1, v_2\}$ on the vertices $V \setminus \{v_1, v_2\}$ is not connected. Let $G_1, \ldots, G_l$, with $l \geq 2$, be the connected components of $G \setminus \{v_1, v_2\}$. For each $i \in \{1, \ldots, l\}$, let G_i^* be the graph induced by $V(G_i) \cup \{v_1, v_2\}$. Observe that G_i^* contains at least one edge incident to v_1 and at least another incident to v_2.

▶ **Theorem 3.5.** *Let $\bar{F}$ be a maximal plane subgraph of D_n, $n \geq 3$. Then, for each separation pair v_1, v_2 of $\bar{F}$, at least one of the components $\bar{F}_i^*$ must be 2-connected.*

Proof. Suppose that v_1, v_2 is a separation pair of $\bar{F}$, that $\bar{F} \setminus \{v_1, v_2\}$ has the connected components $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_l$, and that none of the components $\bar{F}_i^*$ is 2-connected. Then each subgraph $\bar{F}_i^*$ contains at least one cut vertex u_i. Since $\bar{F}_i$ is connected and there exist edges in $\bar{F}_i^*$ incident to v_1 and v_2, vertex u_i is different from v_1 and v_2. The graph $\bar{F}_i^* \setminus \{u_i\}$ has exactly two components, one containing v_1 and the other containing v_2, as otherwise $\bar{F}$ would not be 2-connected. Thus, any path in graph $\bar{F}_i^*$ from v_1 to v_2 must use u_i. In particular, v_1v_2 cannot be an edge of $\bar{F}$. Besides, since v_1, v_2 are in different connected components of $\bar{F}_i^* \setminus \{u_i\}$, if R is the face of $\bar{F}_i^*$ where point u_i appears at least twice, then any continuous curve connecting v_1 to v_2 either contains u_i or some point of the interior of R.

Since $\bar{F}$ is 2-connected, graph $\bar{F} \setminus \{v_1\}$ is connected with v_2 as a cut vertex. As $\bar{F}$ is plane, we can suppose, w.l.o.g., that v_1 is in the outer face of $\bar{F} \setminus \{v_1\}$ and that around vertex v_1 clockwise first there appear the edges to some vertices of component $\bar{F}_1$, then edges connecting v_1 to points of $\bar{F}_2$ and so on. See Figure 4. Therefore v_1 and v_2 must be in the faces R_i of $\bar{F}$ placed between the last edge from v_1 to $\bar{F}_i$ and the first edge from v_1 to $\bar{F}_{i+1}$, for $i \in \{1, \ldots, l\}$. As, by maximality, no edge is entirely in any of those R_i faces, Lemma 3.4 implies that no point of the edge v_1v_2 in D_n can be inside any R_i. Thus, v_1v_2 must begin between two edges v_1v, v_1v' with both v and v' belonging to a common connected component $\bar{F}_i$. However, since u_i belongs to the faces R_{i-1} and R_i, any curve from v_1 and v_2 passes through point u_i or through the interior of R_{i-1} or R_i. Therefore, $\bar{F}$ cannot be maximal. ◀

We call a graph *essentially 3-edge-connected* if it stays connected after removing any two edges not sharing a vertex of degree 2 (i.e., the graph either stays connected or one component is a single vertex). Theorem 3.5 implies that a maximal plane subgraph is essentially 3-edge-connected: If the removal of two edges v_1v_2 and $v_1'v_2'$ results in two non-trivial components (see Figure 5), then the separation pair v_1, v_2' gives no 2-connected component.

▶ **Corollary 3.6.** *Any maximal plane subgraph of a simple topological drawing of K_n is essentially 3-edge-connected.*

Finally, we mention another interesting implication of Theorem 3.1; for a vertex v, we can augment $S(v)$ to a 2-connected plane graph, and thus the remaining part contains a tree.

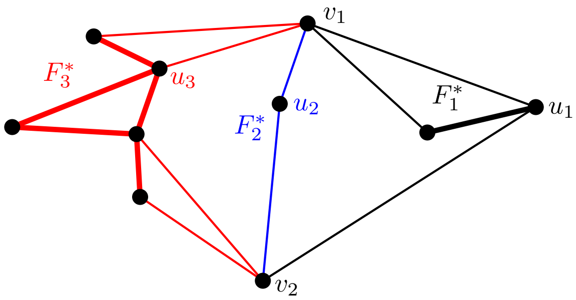

Figure 4 A plane graph with separating pair v_1, v_2 and three components F_i^*, none 2-connected.

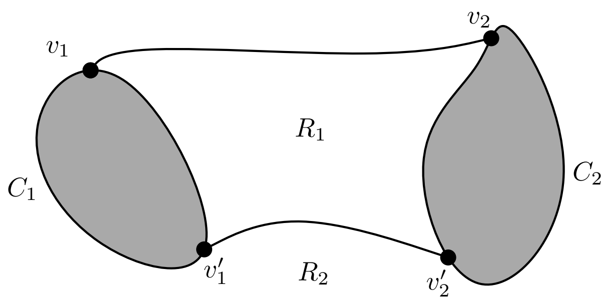

Figure 5 A graph that is not essentially 3-edge-connected. The separation pair v_1, v_2' gives two components $C_1 \cup \{v_1'v_2'\}$ and $C_2 \cup \{v_1v_2\}$, neither of which is 2-connected.

▶ **Corollary 3.7.** *Let $S(u)$ be the edges of D_n incident to a vertex u. There exist a tree T_u spanning the vertices $V \setminus \{u\}$, such that the edges of $S(u) \cup T_u$ form a plane subgraph of D_n.*

▶ **Open Problem 1.** *Given a not necessarily connected plane graph F in D_n, plus a vertex v not in F, can the edges of $S(v)$ incident to but not crossing F be found in $o(n^2)$ time?*

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