A Note on Universal Point Sets for Planar Graphs*

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Abstract

We investigate which planar point sets allow simultaneous straight-line embeddings of all planar graphs on a fixed number of vertices. We first show that at least $(1.293 - o(1))n$ points are required to find a straight-line drawing of each n-vertex planar graph (vertices are drawn as the given points); this improves the previous best constant 1.235 by Kurowski (2004).

Our second main result is based on exhaustive computer search: We show that no set of 11 points exists, on which all planar 11-vertex graphs can be simultaneously drawn plane straight-line. This strengthens the result by Cardinal, Hoffmann, and Kusters (2015), that all planar graphs on $n \leq 10$ vertices can be simultaneously drawn on particular "universal" sets of n points while there are no universal sets of size $n \geq 15$. Moreover, we provide a set of 23 planar 11-vertex graphs which cannot be simultaneously drawn on any set of 11 points. This, in fact, is another step towards a (negative) answer of the question, whether every two planar graphs can be drawn simultaneously – a question from Brass, Cenek, Duncan, Efrat, Erten, Ismailescu, Kobourov, Lubiw, and Mitchell (2007).

1 Introduction

A point set S in the Euclidean plane is called *n-universal for* a family $\mathcal{G}$ of planar n-vertex graphs if every graph G from $\mathcal{G}$ admits a plane straight-line embedding such that the vertices are drawn as points from S. A point set, which is n-universal for the family of all planar graphs, is simply called *n-universal*. We denote by $f_p(n)$ the size of a minimal n-universal set (for planar graphs), and by $f_s(n)$ the size of a minimal n-universal set for stacked triangulations, where stacked triangulations (a.k.a. planar 3-trees) are defined as follows:

▶ **Definition 1.1** (Stacked Triangulations)**.** Starting from a triangle, one may obtain any stacked triangulation by repeatedly inserting a new vertex inside a face (including the outer face) and making it adjacent to all the three vertices contained in the face.

Figures 2 and 3 show examples of stacked triangulations on 11 vertices.

De Fraysseix, Pach, and Pollack [10] showed that every planar n-vertex graph admits a straight-line embedding on a $(2n - 4) \times (n - 2)$ grid – even if the combinatorial embedding is prescribed. Moreover, the graphs are only embedded on a triangular subset of the grid. Hence, $f_p(n) \leq n^2 - O(n)$. This bound was further improved to the currently best known bound $f_p(n) \leq \frac{n^2}{4} - O(n)$ [4] (cf. [19, 5]). Also various subclasses of planar graphs have been studied intensively: Any stacked triangulation on n vertices (with a fixed outer face) can be drawn on a particular set of $f_s(n) \leq O(n^{3/2} \log n)$ points [13]. The first lower bound on the size of n-universal sets substantially greater than n was also given by de Fraysseix, Pach, and Pollack [10], who showed a lower bound of $f_p(n) \geq n + (1 - o(1))\sqrt{n}$. This was further

* The full version is available online [18].

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improved by Chrobak and Karloff [9], and later on Kurowski [16] obtained the previous best lower bound of $(1.235 - o(1))n$ for $f_s(n)$ and thus also $f_p(n)$.

Cardinal, Hoffmann, and Kusters [8] showed that n-universal sets of size n exist for every $n \leq 10$, whereas for $n \geq 15$ no such set exists – not even for stacked triangulations. Moreover, they found a collection of 7,393 planar graphs on $n = 35$ vertices which cannot be simultaneously drawn straight-line on a common set of 35 points. We call such a collection of graphs a *conflict collection*. This was a first big step towards an answer to the question by Brass and others [6], which can be reformulated as follows:

▶ Question 1. Is there a conflict collection of size 2?

2 Results

Our first result is the following theorem, which further improves the lower bound on $f_s(n)$. We present the sketch of the proof in Section 3; for a detailed proof, see the full version [18].

▶ **Theorem 2.1.** *It holds that $f_s(n) \geq (\alpha - o(1))n$, where $\alpha = 1.293\ldots$ is the unique real-valued solution of the equation $\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} = 2$.*

In Section 4 we present our second result, which is another step towards a (negative) answer of Question 1 and strengthens the results from [8]. Its proof is based on exhaustive computer search.

▶ **Theorem 2.2** (Computer-assisted)**.** *There is a conflict collection consisting of 23 stacked triangulations on 11 vertices. Furthermore, there is no conflict collection consisting of 16 triangulations on 11 vertices.*

▶ **Corollary 2.3.** *There is no 11-universal set of size 11 – even for stacked triangulations. Hence, $f_p(11) \geq f_s(11) \geq 12$.*

3 Proof of Theorem 2.1

To prove the theorem, we use a refined counting argument based on a construction of a set of labeled stacked triangulations that was already introduced in [8]. There it was used to disprove the existence of n-universal sets of $n \geq 15$ points for the family of stacked triangulations.

▶ **Definition 3.1** (Labeled Stacked Triangulations, cf. [8, Section 3])**.** For every integer $n \geq 4$, we define the family $\mathcal{T}_n$ of labeled stacked triangulations on the set of vertices $V_n := \{v_1, ..., v_n\}$ inductively as follows:

(i) $\mathcal{T}_4$ consists only of the complete graph K_4 with labels $v_1, \ldots, v_4$.

(ii) If T is a labeled graph in $\mathcal{T}_{n-1}$ with $n \geq 5$, and $v_i v_j v_k$ defines a face of T, then the graph obtained from T by stacking the new vertex v_n to $v_i v_j v_k$ (i.e., connecting it to v_i, v_j, and v_k) is a member of $\mathcal{T}_n$.

The following, which is a consequence of Lemmas 1 and 2 in [8], is the basis of the proof of the new lower bound.

▶ **Corollary 3.2.** *The following two statements hold:*

(i) For any $n \geq 4$, $\mathcal{T}_n$ contains exactly $2^{n-4}(n-3)!$ stacked triangulations.

(ii) Let $P = \{p_1, \ldots, p_m\}$ be a set of $m \geq n \geq 4$ labeled points in the plane. Then for any injection $\pi : V_n \to P$, there is at most one $T \in \mathcal{T}_n$ such that the embedding of T, which maps each vertex v_i to the point $\pi(v_i)$, defines a straight-line-embedding of T.

Sketch of Proof for Theorem 2.1. Let $n \geq 4$ be arbitrary and $m := f_s(n) \geq n$. There exists an n-universal point set $P = \{p_1, \ldots, p_m\}$ for all stacked triangulations, hence for every $T \in \mathcal{T}_n$ there exists a straight-line embedding of T on P, with (injective) vertex-mapping $\pi : V_n \to P$. By Corollary 3.2 (ii), we know that no two stacked triangulations from $\mathcal{T}_n$ (each of which has the same vertex set) yield the same injection π. We conclude that

$$2^{n-4}(n-3)! = |\mathcal{T}_n| \leq \frac{m!}{(m-n)!},$$

Reformulating this inequality using Stirling's approximation now yields with $\beta(n) := \frac{f_s(n)}{n}$

$$2 - o(1) \leq \frac{\beta(n)^{\beta(n)}}{(\beta(n)-1)^{\beta(n)-1}}.$$

Consequently, $\beta(n) \geq (1-o(1))\alpha$, where α is the unique real-valued solution to $\frac{\alpha^\alpha}{(\alpha-1)^{\alpha-1}} = 2$. This proves $f_s(n) = n \cdot \beta(n) \geq (1-o(1))\alpha n$, which is the claim. ◀

4 Proof of Theorem 2.2 and Corollary 2.3

In the following, we outline the strategy which we have used to find a conflict collection of 23 stacked 11-vertex triangulations. Some details are omitted in this extended abstract but can be found in the full version [18]. In particular, we there provide detailed descriptions of all our programs – source codes are available on our supplemental website [17].

It is not hard to see that the embeddability of a given planar graph on a point set does not depend on the exact positions of the points but only on its *order type*, which is a combinatorial encoding of the point set determined by the orientations of triples of points in the point set. Thus, when testing for universality, it suffices to check embeddability of the corresponding graphs only on one representative point set for each order type.

4.1 Enumeration of Order Types

The database of all order types of up to $n = 11$ points was developed by Aurenhammer, Aichholzer, and Krasser [2, 3] (see also Krasser's dissertation [15]). The file for all order types of up to $n = 10$ points (each represented by a point set) is available online, while the file for $n = 11$ requires almost 100GB of storage and is available on demand [1]. In the full version, we also present an alternative and independent approach to enumerate all abstract order types from scratch and provide the corresponding source code [17].

4.2 Enumeration of Planar Graphs

To enumerate all non-isomorphic maximal planar graphs on 11 vertices (i.e, triangulations), we have used the plantri graph generator [7]. For various computations on graphs, such as filtering stacked triangulations, we have used SageMath [20].

4.3 Deciding Universality using a SAT Solver

For a given point set S and a planar graph $G = (V, E)$ we model a propositional formula in conjunctive normal form (CNF) which has a solution if and only if G can be embedded on S.

We have used variables to describe the vertex-to-point mapping and variables to describe whether the straight-line segments are "active" in a drawing. It is not hard to use clauses to assert that such a vertex-to-point mapping is bijective. Also it is easy to assert that, if two adjacent vertices u and v are mapped to points p and q, then the straight-line segment pq is active. For each pair of crossing straight-line segments pq and rs (dependent on the order type of the point set) at least one of the two segments is not allowed to be active.

We have implemented a C++ routine which, given a point set and a graph as input, creates an instance of the above described model and then uses the solver MiniSat [11] (see also [12]) to decide whether the graph admits a straight-line embedding.

4.4 Finding Conflict Collections – A Quantitive Approach

Before we actually tested whether a set of 11 points is 11-universal or not, we discovered a few necessary criteria for the point set, which can be checked much more efficiently. These considerations allowed a significant reduction of the total computation times.

Phase 1: Obviously, 11-universal point sets – if they exist – have to have triangular convex hulls. Secondly, the planar graph depicted in Figure 1 asserts an 11-universal set S to have a certain structure. Using these and a couple of other properties not mentioned here, only 293,114,696 of the 2,343,203,071 abstract order types on 11 points remain as candidates.

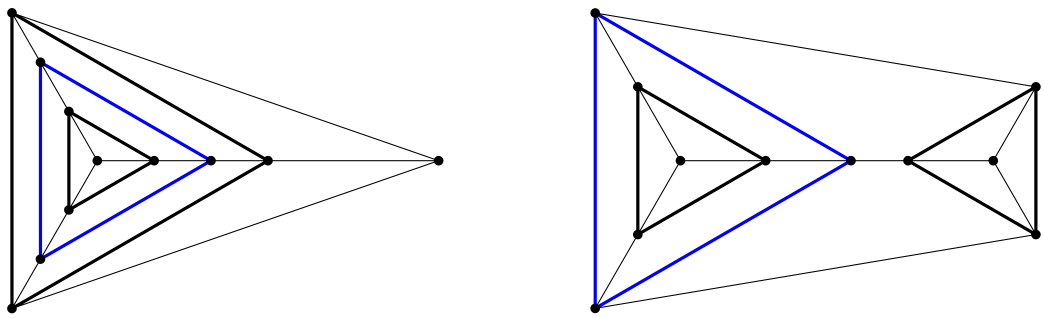

Figure 1 The two embeddings of a graph, which force the point set to have a certain layering.

Phase 2: For each of the remaining order types on 11 points from Phase 1, we have tested the embeddability of all maximal planar graphs on n vertices separately using a SAT-solver based approach. To speed up the computations we have used a priority queue: a graph which does not admit an embedding gets increased priority for other point sets to be tested first.

To keep the conflict collection as small as possible, we first filtered out all point sets which do not allow a simultaneous embedding of all planar graphs on 11 vertices with maximum degree 10. Only 278,530 of the 293,114,696 abstract order types remained (computation time about 100 CPU days).

At this point one can check with only a few CPU hours that the remaining 278,530 abstract order types are not 11-universal. Moreover, since some stacked triangulations on 11 vertices (e.g. the first graph from Figure 2) contain the graph from Figure 1 as a subgraph, the statement even applies to stacked triangulations and Corollary 2.3 follows.

Phase 3: We continued by testing the embeddability for each of the 434 stacked triangulations and each of the 278,530 remaining abstract order types (additional 35 CPU days). Based on this binary information, we formulated an integer program searching for a minimal set of triangulations without simultaneous embedding. Using the Gurobi solver [14], we managed to find a collection $\mathcal{G}$ of 11 stacked triangulations which cannot be embedded simultaneously; see Figure 2. By joining those stacked triangulations to the ones used in Phases 1 and 2, one already obtains a conflict collection of size 95.

Phases 4: To obtain smaller conflict collections, we again repeat the strategy from Phase 2, except that we test for the embeddability of the 11 stacked triangulations from the collection $\mathcal{G}$ obtained in Phase 3 instead of the 82 maximal planar graphs on 11 vertices with maximum degree 10. After 230 CPU days, our program had filtered out 17,533 of the 293,114,696 abstract order types obtained in Phase 1.

Phases 5: We proceeded as in Phase 3 and tested for each of the 434 stacked triangulations and each of the 17,533 order types from Phase 4, whether an embedding is possible (only 2 CPU days). Using the Gurobi solver, we managed to find a collection $\mathcal{H}$ of 12 stacked triangulations, which cannot be simultaneously embedded on those order types; see Figure 2.

Together with the 11 stacked triangulations from $\mathcal{G}$ we obtain a conflict collection of size 23, and the first part of Theorem 2.2 follows.

Phases 6: We have repeated our computations for the union of the two sets of point sets obtained in Phase 3 and Phase 5, respectively, in order to also improve the lower bounds. Using Gurobi, we obtained that any conflict collection consisting of 11-vertex planar graphs has size at least 17. This completes the proof of the second part of Theorem 2.2.

5 Discussion

In Section 3, we provided an improved lower bound for $f_p(n)$ and $f_s(n)$. However, the best known general upper bounds remain far from linear.

One could further proceed with the strategy from Section 4 to find even smaller conflict collection (if such exist). Also one could simply test whether all elements from the conflict collection are indeed necessary, or whether certain elements can be removed.

We also adapted our program to find all n-universal order types on n points for every $n \leq 10$, and hence could verify the results from [8, Table 1].

Unfortunately, we do not have an inductive argument for subsets/supersets of n-universal point sets, and thus the question for $n = 12, 13, 14$ remains open. However, based on computational evidence (see also [8, Table 1]), we strongly conjecture that no n-universal set of n points exists for $n \geq 11$.

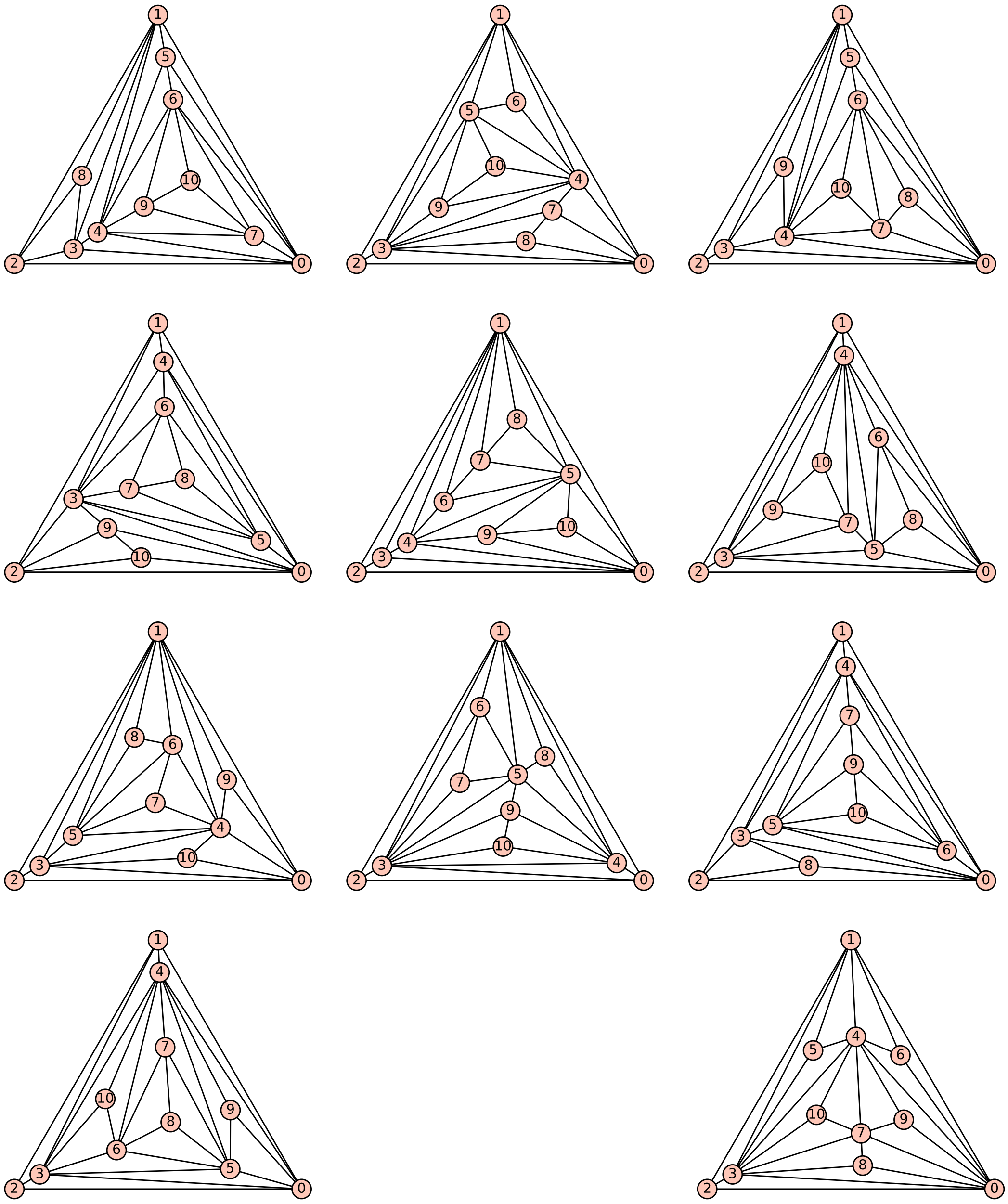

Figure 2 The 11 stacked triangulations from the conflict collection $\mathcal{G}$ obtained in Phase 3.

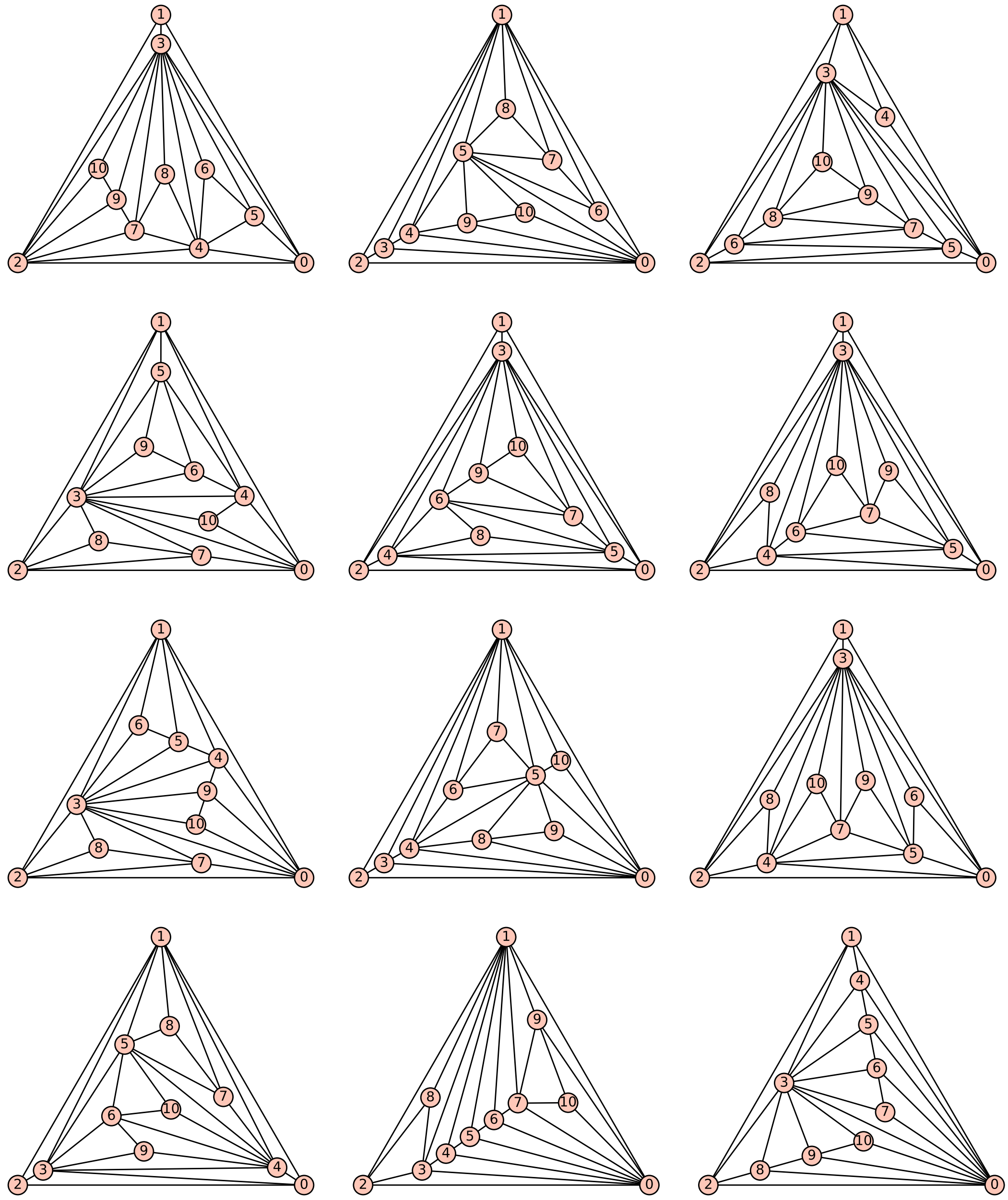

Figure 3 The 12 stacked triangulations from the conflict collection $\mathcal{H}$ obtained in Phase 5.

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