Consistent Digital Curved Rays

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Abstract

Representing a family of geometric objects in the digital world where each object is represented by a set of pixels is a basic problem in graphics and computational geometry. One important criterion is the consistency, where the intersection pattern of the objects should be consistent with axioms of the Euclidean geometry, e.g., the intersection of two lines should be a single connected component. Previously, the set of linear rays and segments has been considered. In this paper, we extend this theory to families of digital curves going through the origin.

1 Introduction

In geometric computation, we often experience that finite-precision computation causes geometric inconsistency. This is because the representation of geometric objects in the pixel world does not always satisfy geometric properties such as Euclidean axioms. Figure 1 shows that a naive definition of digital lines may cause inconsistency, where the intersection of a pair of digital lines has more than one connected components.

Thus, it is important to seek for a digital represention of a family of geometric objects such that they satisfy a digital version of geometric properties. We propose the *consistent digital curved rays* in this paper, generalising consistent digital rays for straight lines [1, 4].

We consider the triangular region Δ defined by $\{(x, y) : x \geq 0, y \geq 0, x + y \leq n\}$ in the plane, and the integer grid $G = \{(i, j) : i, j \in \{0, 1, \ldots, n\}, i + j \leq n\}$ in the region. We can also handle a square region, but use Δ for ease of description of our method.

Each element of G is called a pixel (usually, a pixel is a square, but we represent it by its lower-left-corner grid point in this paper). We say a pixel is a boundary pixel if it lies on $x + y = n$. We consider an undirected graph structure under the four-neighbor topology such that $(i, j) \in G$ is connected to $(k, \ell) \in G$ if $(k, \ell) \in \{(i - 1, j), (i, j - 1), (i + 1, j), (i, j + 1)\}$.

A *digital ray* $S(p)$ is a path in G from the origin o to p, where $S(o) = \{o\}$ is a zero-length path. A family $\{S(p) : p \in G\}$ of digital rays uniquely assinged to each pixel is called *consistent* if the following three conditions hold:

1. If $q \in S(p)$, then $S(q) \subseteq S(p)$.
2. For each $S(p)$, there is a (not necessarily unique) boundary pixel r such that $S(p) \subseteq S(r)$.
3. Each $S(p)$ is a shortest path from o to p in G.

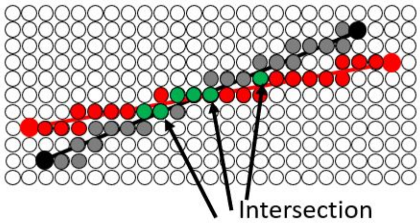

Figure 1 Inconsistency of intersection (green pixels) of two digital line segments

35th European Workshop on Computational Geometry, Utrecht, The Netherlands, March 19–20, 2019.
This is an extended abstract of a presentation given at EuroCG'19. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

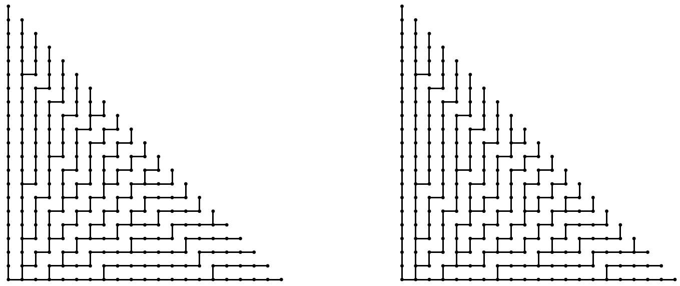

Figure 2 Consistent digital rays simulating linear rays (left) and parabolic rays(right)

The third condition is sometimes omitted in the literature, since it is not suitable for other types of grids such as a triangular grid, but we include it in this paper. The consistency implies that the union of paths $S(p)$ form a spanning tree T in G such that all leaves are boundary pixels as shown in Figure 2, and accordingly the intersection of two digital rays consists of single connected component. The tree T and also the family of digital rays are called CDR (*Consistent Digital Rays*).

Previously, the theory has been considered only for digital straightness [3]. Lubby [4] first gave a construction of CDR so that each $S(p)$ simulates a linear ray within Hausdorff distance $O(\log n)$, and showed that it is asymptotically tight. The construction was re-discovered by Chun *et al.*[1] to give further investigation, and Christ *et al.*[2] gave a construction of consistent digital line segments where the lines need not go through the origin.

We will extend the theory to families of curves with the same topology as linear rays.

A family $\mathcal{F}$ of nondecreasing curves in Δ is called *ray family* if each curve goes through the origin o, and for each point $(x, y) \in \Delta \setminus \{o\}$ there exists a unique curve of $\mathcal{F}$ going through it. We call an element of $\mathcal{F}$ a *ray*. Accordingly, each pair of rays intersect each other only at the origin. A typical example is the family of parabolas $y = ax^2$ for $a \geq 0$.

We give a construction method of CDR $T_{\mathcal{F}}$ in G such that the (unique) ray of $\mathcal{F}$ connecting o and a pixel p is simulated by the path $S(p)$ well, and show an $O(\sqrt{n \log n})$ bound of the Hausdorff distance for several ray families. Although the theoretical bound is much worse than the $\Theta(\log n)$ bound for the linear ray, it is the first nontrivial result for curved rays as far as the authors know, and experimentally the construction works better.

2 Consistent digital rays and their properties

Let us consider a CDR T of G. The set of pixels of G on the diagonal $x + y = k$ for $k = 0, 1, \ldots, n$ is called the level set $L(k)$. We call an edge from $L(k-1)$ towards $L(k)$ an incoming edge to (resp. outgoing edge from) a node in the level $L(k)$ (resp. $L(k-1)$). The following observation was given by Chun *et al.*[1](see Figure 3 for its illustration).

▶ **Lemma 2.1.** *In the level set $L(k)$ for $k \geq 1$, there exists a real value $0 < x(k) \leq k$ such that incoming edge of T to each node whose x-value is smaller than (resp. larger than or equal to) $x(k)$ is vertical (resp. horizontal). Accordingly, there exists a unique branching node of T in $L(k-1)$ (colored yellow in Figure 3).*

Thus, a CDR is completely characterized by the integer sequence $\lceil x(1) \rceil, \lceil x(2) \rceil, \ldots, \lceil x(n) \rceil$, where $1 \leq x(i) \leq i$. The following lemma is easy to verify.

▶ **Lemma 2.2.** *A (unique) CDR exists for each of $(n-1)!$ possible sequences as above.*

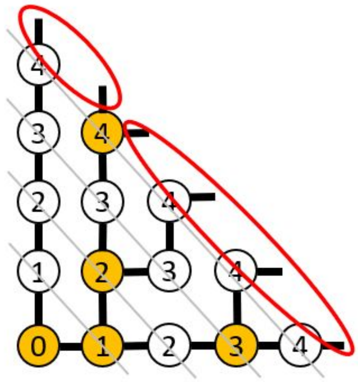

Figure 3 The branching nodes (colored yellow) and partition of incoming edges to the 5th level.

2.1 CDR for linear rays revisited

The CDR of linear rays given by Chun *et al.*[1] can be obtained by selecting $x(k)$ as uniformly as possible from $[1, k]$ by using a low-discreapancy pseudorandom sequence.

Let us consider the binary representation $k = \sum_{i=0}^{\infty} a(i)2^i$ of a natural number k. The van der Courput sequence (see [5]) is the sequence defined by a function $V(k) = \sum_{i=1}^{\infty} a(i)2^{-i}$ from natural numbers to $[0, 1]$. We remove $V(0) = 0$ from our consideration so that the range becomes $(0, 1]$. For example, for $6 = 2 + 4 = 110_2$, $V(6) = 0.11_2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$, where a sequence with a subscript 2 means the 2-adic representation of numbers.

The van der Courput sequence is known to be a low discrepancy sequence as shown in the following lemma (see e.g. [5]).

▶ **Lemma 2.3.** *Consider the set of points $S = \{k, V(k) : k = 0, 1, 2, \ldots, n\}$ in the region $X = [0, n] \times [0, 1]$. Then, for any axis parallel rectangle R in X, the difference from the number of points in $S \cap R$ and the area of R is $O(\log n)$.*

In particular, for each $m < n$, the set $\{V(i) : m \leq i \neq n\}$ gives an almost uniform distribution on $[0, 1]$ deterministically. We can set $x(k) = kV(k)$ to obtain a CDR that approximates the linear rays emanating from the origin with $O(\log n)$ distance bound. In order to generalize to the curved rays, we give the following interpretation.

Consider a line $y = ax$ intersecting $x + y = k$ at $q = (x_0, k - x_0)$. By definition, its slope is a, which is $\frac{k-x_0}{x_0}$. Naturally, we want to draw the line in the neighborhood of q with a segment of slope $\frac{k-x_0}{x_0}$, but we need to approximate it with a grid path. Therefore, ideally the ratio of vertical edges to the horizontal edges in the paths should be $\frac{k-x_0}{x_0}$ in a neighborhood of q.

By the definition of $x(k)$, the edge incoming to q is vertical if and only if q lies on the left of $x(k)$. If we take $x(k) = kV(k)$, the probability[1] that q is to the left of $x(k)$ is $\frac{x_0}{k}$, since $V(k)$ gives a uniform distribution. Thus, the incoming edge becomes horizontal and vertical with probabilities $\frac{x_0}{k}$ and $\frac{k-x_0}{k}$, respectively. Hence, the ratio between them is $\frac{k-x_0}{x_0}$ as desired.

We would like to extend this argument for other families of curves.

[1] Since the process is deterministic, we should say "ratio" rigorously, but we use the term "probability" for convenience' sake.

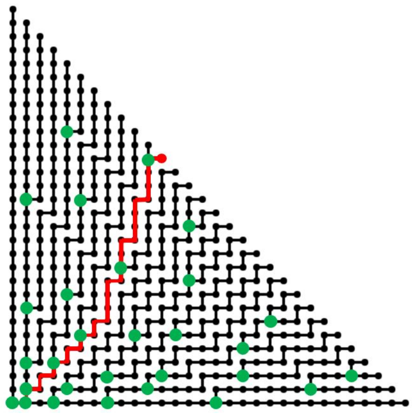

Figure 4 CDR T_{para}. Green nodes are branching nodes. Red path gives a digital parabolic ray.

3 CDR for families of curves

3.1 CDR for a family of parabolas

To improve readability, we start with the ray family $y = ax^2$ $(a \geq 0)$ of parabolas. We include the y-axis $x = 0$ in the family (this convention is applied to all other cases).

Consider a parabola $y = ax^2$ intersecting the level $x+y = k$ at $q = (x_0, k - x_0)$, The slope of tangent at q is $2ax_0$, which is $\frac{2(k-x_0)}{x_0}$. In order to approximate the parabola nicely, the tangent segment in the neighborhood of q should be approximated by a path that contains the horizontal edge with probability $\frac{x_0}{2k-x_0}$.

Thus, we should select $x(k)$ to be located on the left of q with probability $\frac{x_0}{2k-x_0}$. If we set $x_0 = kt$, this probability equals $\frac{t}{2-t}$. We consider a monotonically increasing function F in the range $[0, 1]$ and set $x(k) = kF(V(k))$. The probability that $x_0 = kt < x(k)$ is the probability that $F^{-1}(t) < V(k)$ from the monotonicity of F. Because of uniformity of $V(k)$, this probability equals $F^{-1}(t)$ (ignoring the small discrepancy).

Then, the probability (over k) that q is on the left of $x(k)$ is same as $t = x(k)/k \leq F(V(k))$. This probability is same as the probability that $F^{-1}(t) \leq V(k)$ from the monotonicity of F. Because of uniformity of $V(k)$, this means $F^{-1}(t) = \frac{t}{2-t}$ to meet our requirement, and $F(z) = \frac{2z}{z+1}$. Thus, we have $x(k) = \frac{2kV(k)}{V(k)+1}$ to define a CDR T_{para} illustrated in Figure 4.

The following theorem ensures that T_{para} approximate parabola rays well theoretically, and we will also demonstrate it works even better by implementation later.

▶ **Theorem 3.1.** *For each node $p = (i, j) \in G$, the Hausdorff distance between the parabola ray going through p and the path $S(p)$ from p towards the origin in T_{para} is $O(\sqrt{n \log n})$.*

The theorem is derived from the following lemma, which is obtained from Lemma 2.3. We omit proofs in this version.

▶ **Lemma 3.2.** *Consider the set of points $S = \{(k, V(k)) : k = 0, 1, 2, \ldots, n\}$. Let $f(x)$ be a nonincreasing or nondecreasing continuous function from $[0, n]$ to $[0, 1]$, and let $Q_I(f) = \{(x, y) : 0 \leq y \leq f(x), x \in I\}$ for any given interval $I \subset [0, 1]$. Then, the discrepancy (i.e., difference from the number of points in $S \cap Q_I(f)$ and the area $A(Q_I(f))$) is bounded by $c\sqrt{n \log n}$ for a suitable constant c.*

Note that for the discrepancy discussed in the above lemma, an $\Omega(\sqrt{n})$ lower bound is known even for a linear function [5].

3.2 Homogeneous polynomials

Le us consider the family $\mathcal{F}_j$ of curves defined by $y = f_a(x) = ax^j$ for $a \geq 0$. Here, the slope of the tangent of a curve at (x, y) is jax^{j-1}, which is jy/x. Thus, analogously to the parabola case, we have $F^{-1}(t) = \frac{t}{j-(j-1)t}$ and $F(z) = \frac{jz}{1+(j-1)z}$. We set $x(k) = \frac{jkV(k)}{(j-1)V(k)+1}$ for $k = 1, 2, \ldots, n$ to define a CDR $T_{\mathcal{F}_j}$. The following is obtained analogously to the parabola case.

▶ **Theorem 3.3.** *The path from p to the origin o in the CDR $T_{\mathcal{F}_j}$ approximates the curve in $\mathcal{F}_j$ going through p and o with an $O(\sqrt{n \log n})$ distance bound.*

3.3 Framework for a family of constant-multiplied curves

More generally, let us consider a nondecreasing differentiable function $y = f(x)$ for $x \in [0, n]$ such that $f(0) = 0$ and $f(x) > 0$ for $x > 0$. We define the family $\mathcal{F} = \{C_a : a \geq 0\}$ of curves, where C_a is defined by $y = af(x)$.

If C_a goes through (x_0, y_0), then $a = \frac{y_0}{f(x_0)}$. The slope of the curve C_a at (x_0, y_0) is $af'(x_0)$, which is (eliminating a) $\frac{f'(x_0)}{f(x_0)}(y_0)$. We consider the slope $T(x, k) = \frac{f'(x)}{f(x)}(k - x)$ along the diagonal $x + y = k$ for each k. We assume that it is monotonically decreasing in x for each fixed k.

Thus, we want to control so that the probability that the edge incoming to a pixel $(x, k - x)$ in $L(k)$ is horizontal with probability $\frac{1}{1+T(x,k)}$.

We consider a F such that $x(k) = kF(V(k))$ so that $(x, k - x)$ becomes horizontal with probability $\frac{1}{1+T(x,k)}$. Because of uniformity of $V(k)$, we set $F^{-1}(t) = \frac{1}{1+T(kt,k)}$. Note that we can show that F is monotone and the above argument holds. Although the explicit form of F might not be obtained, we can apply binary search to compute $F(z)$ for a given z utilizing the monotonicity. Thus, we can compute $x(k) = kF(V(k))$ within the pixel precision in $O(\log n)$ time.

3.3.1 Sigmoid curves and sine curves

Let $\mathcal{F}_{sig} = \{a\sigma(x) : 0 \leq a\}$, where $\sigma(x) = \frac{1}{1+e^{-x}} - \frac{1}{2}$ is the shifted sigmoid function. The curves $y = a\sigma(x)$ satisfy our conditions, and we have a CDR with distance bound $O(\sqrt{n \log n})$.

The sine curve $y = \sin(x)$ is not monotone, but we can define $\tilde{\sin}(x)$ by $\tilde{\sin}(x) = 0$ for $x < 0$, $\tilde{\sin}(x) = \sin x$ for $0 \leq x \leq \pi/2$ and $\tilde{\sin}(x) = 1$ for $x > \pi/2$. The curve $y = \tilde{\sin}(x)$ is monotonically nondecreasing and differentiable, and we can apply our CDR construction for the family of curves $y = a\, \tilde{\sin}(x)$ for $a \geq 0$.

The family of logarithmic functions $y = a \log(x + 1)$ can be also similarly handled.

The $O(\sqrt{n \log n})$ distance bound follows analogously to the parabola case for each family. Details are omitted in this version.

Figure 5 illustrates CDR of families discussed above.

4 Experimental result and conclusion

For each grid width $n = 2^m$ up to $n = 2^{14}$, the worst-case Hausdorff distance between parabolas and digital rays in T_{para} is given in Figure 5, where it is about 12 for $n = 2^{14}$.

The experimental result suggests that the distance bound could be polylogarithmic, and our $O(\sqrt{n \log n})$ bound seems to be loose, although currently the lower bound mentioned for Lemma 3.2 prevents us to improve it beyond $\sqrt{n}$. On the other hand, Figure 6 suggests

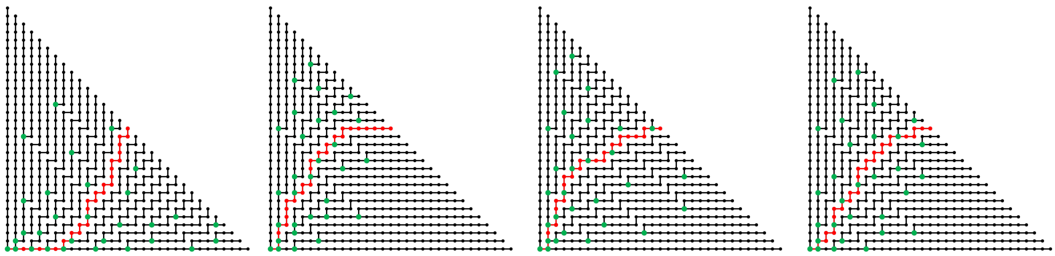

Figure 5 CDRs for (left to right) $y = ax^3$, $y = a\tilde{\sin}x$, $y = a\sigma(x)$, and $y = a\log(x+1)$. Green nodes are branching nodes. Each red path goes to the center boundary node.

the distance bound tends to be slightly larger than $O(\log n)$ for this construction, and investigation on both lower and upper bounds remains an interesting open problem. Another interesting problem is to find construction of consistent digital curves removing the ray condition. For example, it is curious to handle the set of all axis parallel parabolas.

Acknowledgement This work is supported by JSPS Kakenhi 17K19954 and 18H05291.

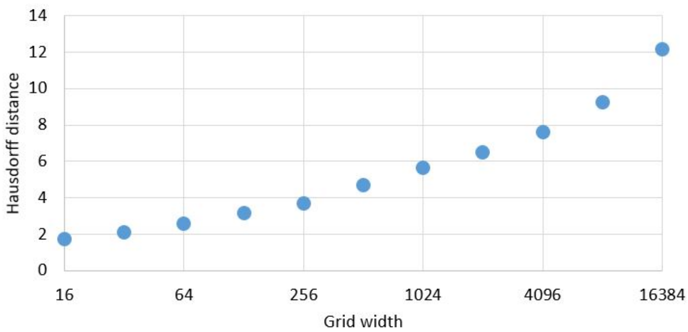

Figure 6 The largest distance from parabola ray and path in T_{para}.

References

1 J. Chun, M. Korman, M. Nöllenburg, T. Tokuyama, Consistent Digital Rays, *Discrete & Computational Geometry* 42-3 (2009) pp.359-378.

2 T. Christ, D. Pálvölgyi, M. Stojaković, Consistent Digital Line Segment, *Discrete & Computational Geometry* 47-4 (2012), pp. 691-710.

3 R. Klette, A. Rosenfeld, Digital straightness – a review *Discrete Applied Math.* 139 (2004), 197-230.

4 M.G.Lubby, Grid geometries which preserve properties of euclidean geometry: A study of graphics line drawing algorithms. In NATO Conference on Graphics/CAD, pages 397-432,1987.

5 J. Matoušěk, *Geometric Discrepancy*, Springer Verlag 1991.