Maximum Rectilinear Convex Subsets *

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Abstract

Let P be a set of n points in the plane. We consider a variation of the classical Erdős-Szekeres problem, presenting efficient algorithms with $O(n^3)$ running time and $O(n^2)$ space which compute: (1) A subset S of P such that the boundary of the rectilinear convex hull of S has the maximum number of points from P, (2) a subset S of P such that the boundary of the rectilinear convex hull of S has the maximum number of points from P and its interior contains no element of P, and (3) a subset S of P such that the rectilinear convex hull of S has maximum area and its interior contains no element of P.

1 Introduction

Let P be a point set in general position in the plane. A subset S of P with k points is called a *convex k-gon* if the elements of S are the vertices of a convex polygon, and it is called a *convex k-hole* if the interior of the convex hull of S contains no point of P. The study of convex k-gons and convex k-holes of point sets started in a seminal paper by Erdős and Szekeres [11]. Since then, a plethora of papers studying both the combinatorial and the algorithmic aspects of convex k-gons and convex k-holes has been published. The reader can consult the two survey papers [8, 12] about so-called Erdős-Szekeres type problems.

Some recent papers studying the existence and the number of convex k-gons and convex k-holes for sets of points in the plane are [1, 2, 3]. Papers dealing with the problem of finding

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largest convex k-gons and convex k-holes are, respectively, Chvátal and Kincsek [10] and Avis and Rappaport [7], which solve these problems in $O(n^3)$ time and $O(n^2)$ space.

In this paper we study Erdős-Szekeres type problems under a variation of convexity known as *rectilinear convexity* (or *orthoconvexity*): Let $P = \{p_1, \ldots, p_n\}$ be a n point set in the plane in general position. A *quadrant* is the intersection of two open half-planes whose supporting lines are parallel to the x- and y-axes. We say that a quadrant Q is *P-free* if it contains no point of P. The *rectilinear convex hull* (or *orthogonal convex hull*) of P, denoted as $RCH(P)$, was introduced by Ottmann et al. [13] (see also [15]) and is defined as:

$$RCH(P) = \mathbb{R}^2 - \bigcup_{Q \text{ is } P\text{-}free} Q.$$

The rectilinear convex hull of a point set might be a simply connected set, yielding an intuitive and appealing structure (see Figure 1, left). However, in other cases the rectilinear convex hull can have several connected components (see Figure 1, right), some of which might be single points which we call *pinched* points. The *size* of $RCH(S)$ is the number of points of S on the boundary of $RCH(S)$. The sizes of the rectilinear convex hulls in Figure 1 are thirteen and twelve. In this paper, we present algorithms for the following problems:

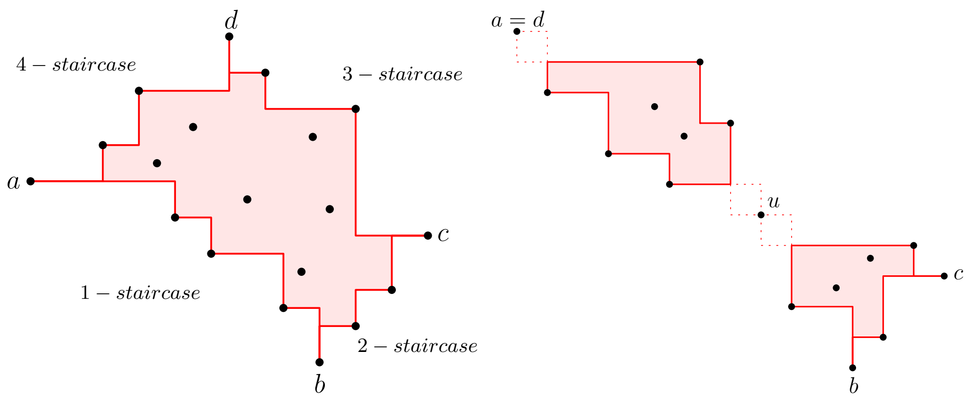

Figure 1 Left: A point set with a connected rectilinear convex hull. Right: A point set whose rectilinear convex hull is disconnected. Two of its components are pinched points.

(1) MaxRCH: Given a set P of n points in the plane, find a subset $S \subseteq P$ such that the size of $RCH(S)$ is maximized. We solve the MaxRCH problem given an algorithm which runs in $O(n^3)$-time and $O(n^2)$-space. Then, we adapt our algorithm to solve the following problems, each in $O(n^3)$ time and $O(n^2)$ space.
(2) MaxEmptyRCH: Given a set P of n points in the plane, find a subset $S \subseteq P$ such that the interior of $RCH(S)$ contains no point of P and the size of $RCH(S)$ is maximized.
(3) MaxAreaRCH: Given a set P of n points in the plane, find a subset $S \subseteq P$ such that the interior of $RCH(S)$ contains no point of P and the area of $RCH(S)$ is maximized.

Related work: Erdős-Szekeres type problems have also been studied for colored point sets. Let P be a set of points such that each of its elements is assigned a color, say red or blue. Bautista-Santiago et al. [9] studied the problem of finding a monochromatic subset S of P of maximum size such that all of the elements of P contained in the convex hull of S have the same color. They also solve the same problem for each element of P having a weight. All of these algorithms run in $O(n^3)$ time and $O(n^2)$ space.

Notation and definitions: For a point p of the plane, let p_x and p_y denote the x- and y-coordinates of p, respectively. For $p \neq q \in \mathbb{R}^2$, we write $p \prec q$ to denote that $p_x < q_x$ and

$p_y < q_y$, and $p \prec' q$ to denote that $p_x < q_x$ and $p_y > q_y$. Any point p in the plane defines four axis-aligned quadrants $Q_i(p)$, $i = 1, 2, 3, 4$ as follows (see Figure 2, left):

- $Q_2(p) = \{q \in \mathbb{R}^2 \mid q \prec' p\}$, $Q_1(p) = \{q \in \mathbb{R}^2 \mid p \prec q\}$,
- $Q_3(p) = \{q \in \mathbb{R}^2 \mid q \prec p\}$, $Q_4(p) = \{q \in \mathbb{R}^2 \mid p \prec' q\}$.

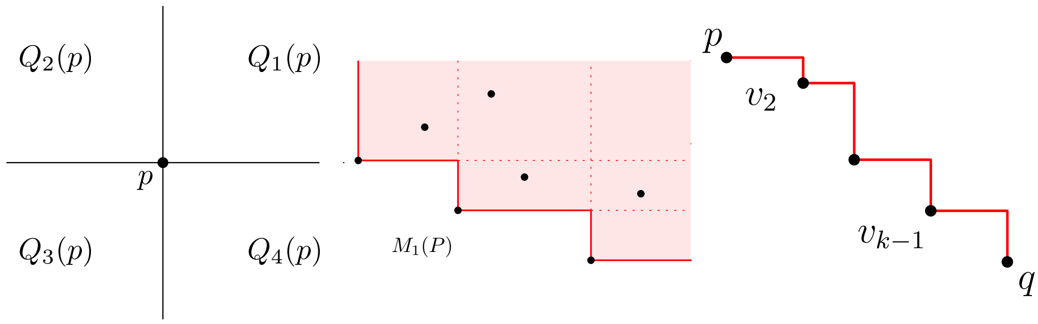

Figure 2 Left: The definition of the sets $Q_i(p)$. Middle: A 7-point set P and the set $M_1(P)$. The vertices of $M_1(P)$ in P are the 1-extremal points of P. Right: A 1-staircase.

Given P, for $i = 1, 2, 3, 4$, let $M_i(P) = \bigcup_{p \in P} \overline{Q_i(p)}$ where $\overline{Q_i(p)}$ denotes the closure of $Q_i(p)$. The elements of P that belong to the boundary of $M_i(P)$, are called the (rectilinear) *i-extremal* points of P (see Figure 2, middle). For every $J \subseteq \{1, 2, 3, 4\}$, we say that $p \in P$ is *J-extremal* if p is j-extremal for every $j \in J$. The *rectilinear convex hull* of P is the set[1]

$$RCH(P) = \bigcap_{i=1}^{4} M_i(P),$$

see Figure 1, left. For the sake of simplicity, we assume that all point sets P considered in this paper are in *general position*, which here means that no two points of P share the same x- or y-coordinate. Let a, b, c, d denote the leftmost, bottommost, rightmost, and topmost points of P, respectively. Note that a is $\{1, 4\}$-extremal, b is $\{1, 2\}$-extremal, c is $\{2, 3\}$-extremal, and d is $\{3, 4\}$-extremal.

i-staircases: Let $S = \{v_1, \ldots, v_k\}$ be a set of vertices such that $v_1 = p$, $v_k = q$, and $v_i \prec' v_j$ for every $i < j$. A 1-*staircase* joining p to q (Figure 2, right) is the boundary of $M_1(S)$ minus the infinite rays (Figure 2, middle), i.e., it is an orthogonal polygonal chain such that two consecutive points of S are joined by an *elbow*, i.e., a horizontal followed by a vertical segment. A 3-*staircase* joining p to q is defined similarly using elbows whose first segment is vertical. Similarly for 2- and 4-staircases, except that we require $v_i \prec v_j$ (Figure 1, left).

The boundary of the rectilinear convex hull of a point set P is a subset of the union of four staircases, a 1-, a 2-, a 3-, and a 4-staircase whose vertices are the 1-, 2-, 3-, and 4-extremal points of P. See again Figure 1. Observe that the rectilinear convex hull $RCH(P)$ of a point set P is disconnected when either the complements $\mathbb{R}^2 - M_1(P)$ and $\mathbb{R}^2 - M_3(P)$ intersect, as shown in Figure 1, right, or the complements $\mathbb{R}^2 - M_2(P)$ and $\mathbb{R}^2 - M_4(P)$ intersect. A pinched point u of $RCH(P)$ occurs when u is either both 1-extremal and 3-extremal, as shown in Figure 1, right, or both 2-extremal and 4-extremal. Note that the *size* of $RCH(P)$ is the number of points of P which are i-extremal for at least one $i \in \{1, 2, 3, 4\}$.

[1] Some recent works [4, 5, 6] use the notation $\mathcal{RH}(P)$ for the rectilinear convex hull of P.

Throughout this paper, we will use a, b, c, and d to denote the leftmost, bottommost, rightmost, and topmost points of P, where a, b, c, d are not necessarily different (see Figure 1, right). Given two points u and v in the plane, let $B(u, v)$ be the smallest open axis-aligned rectangle containing u and v, and let $P(u, v) = P \cap B(u, v)$. We say that $RCH(P)$ is *vertically separable* if $B(a, d)$ and $B(b, c)$ are separated by a vertical line. Given P and a horizontal line ℓ, let P' be the image of P under a reflection around ℓ. The following lemma is key for our algorithms, and we assume, when needed, that P is vertically separable.

▶ **Lemma 1.1.** *$RCH(P)$ or $RCH(P')$ is vertically separable.*

2 Rectilinear convex hull of maximum size

In this section, we solve the MaxRCH problem. For every pair of points p, q such that $p \prec q$, let $\mathcal{C}^i_{p,q}$, $i = 2, 4$, be an i-staircase with endpoints p and q of maximum size. Similarly, if $p \prec' q$, let $\mathcal{C}^i_{p,q}$, $i = 1, 3$, be an i-staircase with endpoints p and q of maximum size, see Figure 3. Our goal is to combine four staircases in order to obtain a subset S of P whose rectilinear convex hull is of maximum size. All of this has to be done carefully, since the occurrence of pinched points may lead to over counting.

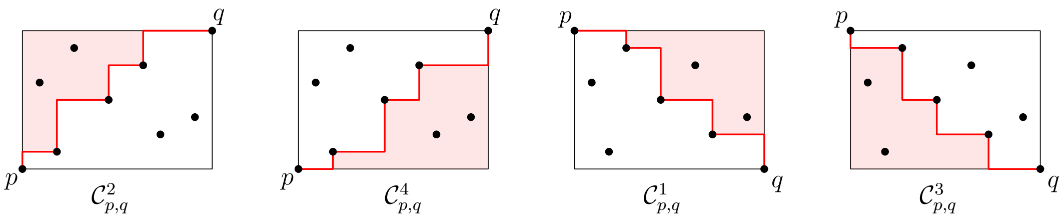

Figure 3 Examples of $\mathcal{C}^i_{p,q}$.

Our algorithm to solve the MaxRCH problem proceeds in three steps: In the first step we calculate all of the $\mathcal{C}^i_{p,q}$, $i = 1, \ldots, 4$. In the second step we calculate what we call *triple staircases* (yet to be defined). In the third step we show how to combine triple staircases and the $\mathcal{C}^4_{p,q}$ staircases to solve the MaxRCH problem. In this step we will make sure that the solution thus obtained is vertically separable. Our algorithm will run in $O(n^3)$ time and $O(n^2)$ space. The main tool is the use of dynamic programming applying to some recurrences. (In the appendix we describe in detail the steps of our algorithm.) Let $C^i_{p,q}$ be the number of elements of P in $\mathcal{C}^i_{p,q}$. Note that $C^i_{p,q}$ equals the maximum number of i-extremal points over all $X \subseteq \{p, q\} \cup P(p, q)$ with $p, q \in X$.

The first step: Compute the numbers $C^i_{p,q}$, for $i \in \{1, 2, 3, 4\}$, $p, q \in P$. These can be done in $O(n^3)$ time and $O(n^2)$ space, using dynamic programming applying the recurrence:

$$C^i_{p,q} = \begin{cases} 1 & \textbf{if } p = q \\ \max\{1 + C^i_{r,q}\} \text{ over all } r \in P(p,q) & \textbf{if } p \neq q. \end{cases} \tag{1}$$

Using the $C^i_{p,q}$, it is a routine matter to determine a staircase $\mathcal{C}^i_{p,q}$ of maximum size.

The second step: Given a point set S, we define the *triple staircase* associated to S, as the concatenation of the 1-, 2-, and the 3-staircases of the rectilinear convex hull of S. In this step, our goal is to obtain triple staircases of maximum cardinality starting and ending at some pairs of points of P. Triple staircases allow us to manage pinched points.

Consider $p, q \in P$ such that $p \prec q$ or $p = q$. Let $Z(p, q) = B(p, q) \cup Q_4(p) \cup Q_4(q)$, and let $z(p, q) = Z(p, q) \cap P$ (see Figure 4). Let S' be a subset of $z(p, q)$ such that the triple staircase, denoted as $\mathcal{T}_{p,q}$, associated to $S' \cup \{p, q\}$ is of maximum cardinality. Observe that $M_1(S') \cap M_2(S') \cap M_3(S')$ may contain points in $P(p, q)$, it may be disconnected, and it may have pinched points. Note that p and q are always the endpoints of $\mathcal{T}_{p,q}$ (see Figure 5). Let $X_{p,q}$ denote the set of extreme vertices of $\mathcal{T}_{p,q}$, and let $T_{p,q}$ be the cardinality of $X_{p,q}$.

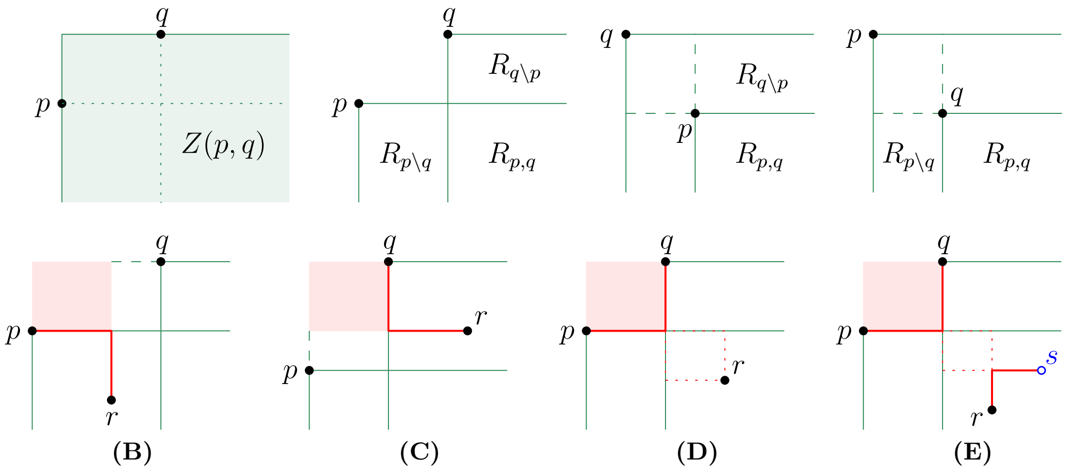

Figure 4 Top: Region $Z(p, q)$ and subsets $R_{p\backslash q}$, $R_{q\backslash p}$, and $R_{p,q}$. Bottom: cases in the recursive computation of $T_{p,q}$.

We calculate all of the $T_{p,q}$'s using Equation (2). Let $\alpha_{p,q} = 1$ if $p = q$, and $\alpha_{p,q} = 2$ if $p \neq q$. We use dynamic programming to compute the values of the table T using the following recurrence:

$$T_{p,q} = \max \begin{cases} C^2_{p,q} & \textbf{(A)} \\ 1 + T_{r,q} \text{ over all } r \in R_{p\backslash q} : P(p, r) = \emptyset & \textbf{(B)} \\ 1 + T_{p,r} \text{ over all } r \in R_{q\backslash p} : P(q, r) = \emptyset & \textbf{(C)} \\ \alpha_{p,q} + T_{r,r} \text{ over all } r \in R_{p,q} : P(p, r) = P(q, r) = \emptyset & \textbf{(D)} \\ \alpha_{p,q} + U_{p,r} \text{ over all } r \in R_{p,q} : P(p, r) = P(q, r) = \emptyset & \textbf{(E)} \end{cases} \tag{2}$$

where

$$U_{p,r} = \max\{T_{r,s}\} \text{ over all } s \in R_{p\backslash r} : P(p, s) = \emptyset. \tag{3}$$

▶ **Lemma 2.1.** *The previous recurrence correctly calculates $T_{p,q}$, the size of $X_{p,q}$.*

We use now triple staircases and the $\mathcal{C}^4_{p,q}$ staircases to solve the MaxRCH problem.

The third step: We proceed as follows. For $a, d \in P$ with $a \prec d$, we compute an optimal solution $S_{a,d} \subseteq P$ having a as its leftmost point and d as its topmost point. Finding $S_{a,d}$ is not as simple as joining the maximum 4-staircase of size $\mathcal{C}^4_{a,d}$ with $\mathcal{T}_{a,d}$, since $\mathcal{T}_{a,d}$ might have extreme vertices in the rectangle $B(a, d)$ which can also be vertices of the 4-staircase. To see how we arrive to this solution, consider a vertically separable optimal solution $S_{a,d}$. We traverse the 1-staircase of $S_{a,d}$ from a to d, and let $e \in S_{a,d}$ be the first point of P that belongs to the set $R_{a,d}$. Let $f \in S_{a,d}$ be the point of P that precedes e in the staircase and belongs to $\{a\} \cup R_{a\backslash d}$ (see Figure 6). Let ℓ be the horizontal line through e. Then, f must satisfy the conditions of the next lemma.

▶ **Lemma 2.2.** *By the optimality of $S_{a,d}$, the point f satisfies: (i) $P(f, e) = \emptyset$, (ii) $C^1_{a,f}$ is maximum among all points of P in $\{a\} \cup R_{a\backslash d}$ that are above ℓ (see Figure 7).*

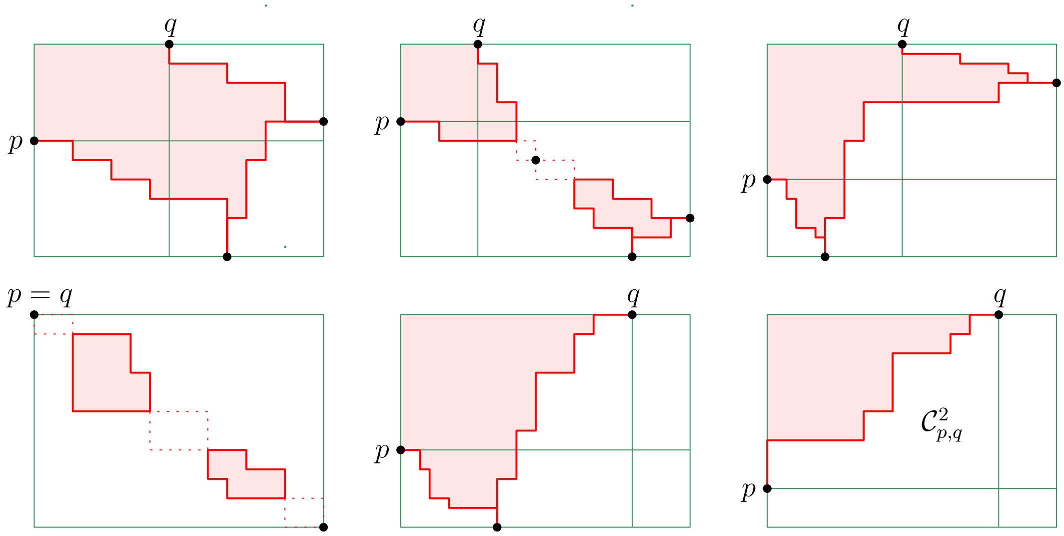

Figure 5 Examples of triple staircases $\mathcal{T}_{p,q}$.

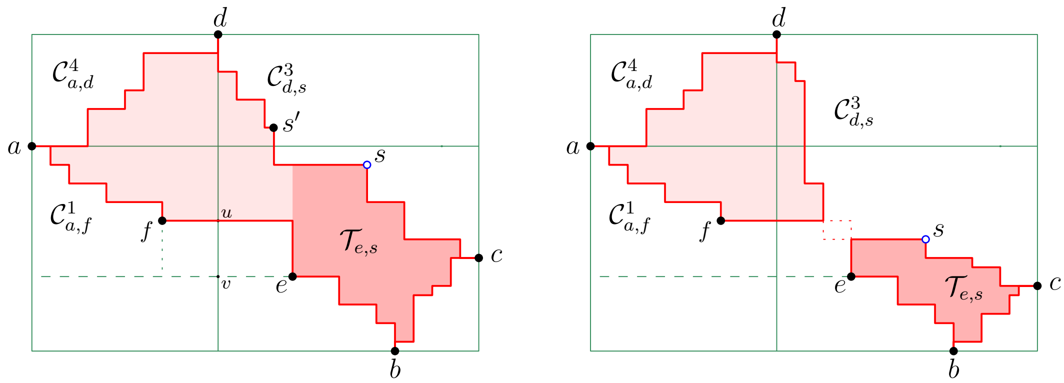

Figure 6 The third step of the algorithm.

Using Lemma 2.2, $S_{a,d}$ can be found as follows. Sweep a line ℓ from top to bottom, stopping at all the points of $\{a\} \cup R_{a\setminus d} \cup R_{a,d}$. Every time ℓ passes over a point of $\{a\} \cup R_{a\setminus d}$, we update the point f satisfying condition (ii) in $O(1)$ time. Furthermore, every time ℓ passes over a point $e \in R_{a,d}$, we verify whether condition (i) is satisfied. If this is the case, we consider e as a candidate to be the first point of the 1-staircase of $S_{a,d}$ in $R_{a,d}$, and set $f(e) = f$. Let E be the set of all points that are candidates to be point e. If $E = \emptyset$, we return $T_{a,d} = -1$. Otherwise, we return:

$$\max_{e \in E} \left\{ C^4_{a,d} + C^1_{a,f(e)} + V_{d,e} - 2 \right\} \tag{4}$$

as the size of the optimal $S_{a,d}$, where table V (similar to table U) is a table such that it contains an entry $V_{d,e}$ for every pair $d, e \in P$ such that $d \prec' e$. Each $V_{d,e}$ satisfies:

$$V_{d,e} \;=\; \max \left\{ C^3_{d,s} + T_{e,s} - 1 \right\} \text{ over all } s \in R_{d\setminus e} \cup \{e\} \text{ such that } pred(s,d) \prec' e \tag{5}$$

where $pred(s,d)$ (s' in Figure 6) is the point of P on $\mathcal{C}^3_{d,s}$ which precedes s while traversing $\mathcal{C}^3_{d,s}$ from d to s. Hence, the size of $RCH(S_{a,d})$ equals:

$$C^4_{a,d} + C^1_{a,f} + C^3_{d,s} + T_{e,s} - 3 \;=\; C^4_{a,d} + C^1_{a,f} + V_{d,e} - 2. \tag{6}$$

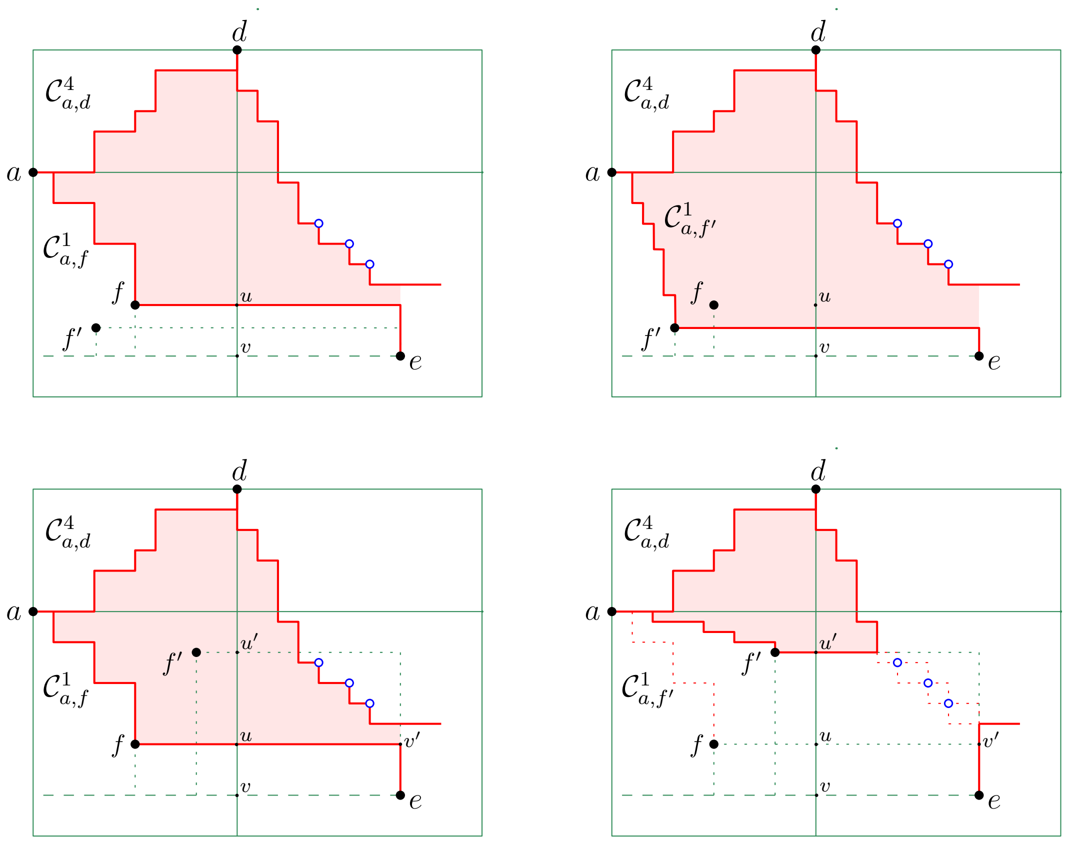

Figure 7 Illustration of Lemma 2.2.

We subtract 3 since a, d, and s are counted twice each. By using table V and Equations (4) and (5), we compute the optimal solution in $O(n^3)$ time and $O(n^2)$ space. Note that all the $pred(s, d)$ can be computed in $O(n^3)$ time using Equation (1). Hence, we have:

▶ **Theorem 2.3.** *The* MaxRCH *problem can be solved in $O(n^3)$ time and $O(n^2)$ space.*

3 Maximum size/area empty rectilinear convex hulls

In this section, we adapt the algorithm of Section 2 to solve the MaxEmptyRCH and the MaxAreaRCH problems. Note that, by Lemma 1.1, we can assume that $B(a, d)$ and $B(b, c)$ are separated by a vertical line in both the MaxEmptyRCH and MaxAreaRCH problems.

To solve the MaxEmptyRCH problem in $O(n^3)$ time and $O(n^2)$ space, we make modifications to the steps of our previous algorithm. The rest of the algorithm is the same.

▶ **Theorem 3.1.** *The* MaxEmptyRCH *problem can be solved in $O(n^3)$ time and $O(n^2)$ space.*

Given a bounded set $Z \subset \mathbb{R}^2$, let $\mathsf{Area}(Z)$ denote the area of Z. To solve the MaxAreaRCH problem, we will proceed as in the previous result, but we need to sum areas, not to count points in all of our recurrences. Given an empty i-staircase $\mathcal{C}^i_{p,q}$, let $C^i_{p,q}$ be now $\mathsf{Area}(B(p, q) \cap M_i(P(p, q)))$.

▶ **Theorem 3.2.** *The* MaxAreaRCH *problem can be solved in $O(n^3)$ time and $O(n^2)$ space.*

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