Delaunay triangulations of symmetric hyperbolic surfaces

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Abstract

Of the several existing algorithms for computing Delaunay triangulations of point sets in Euclidean space, the incremental algorithm has recently been extended to the Bolza surface, a hyperbolic surface of genus 2. We will generalize this algorithm to so called symmetric hyperbolic surfaces of arbitrary genus. Delaunay triangulations of point sets on hyperbolic surfaces can be constructed by using the fact that such point sets can be regarded as periodic point sets in the hyperbolic plane. However, one of the main issues is then that the result might contain 1- or 2-cycles, which means that the triangulation is not simplicial. As the incremental algorithm that we use can only work with simplicial complexes, this situation must be avoided.

In this work, we will first compute the systole of the symmetric hyperbolic surfaces, i.e., the length of the shortest non-contractible loop. The value of the systole is used in a condition to ensure that the triangulations will be simplicial. Secondly, we will show that it is sufficient to consider only a finite subset of the infinite periodic point set in the hyperbolic plane. Finally, we will algorithmically construct a point set with which we can initialize the algorithm.

1 Introduction

The incremental algorithm, one of the known algorithms for computing Delaunay triangulations of point sets in Euclidean space, inserts the points one by one and updates the triangulation after each insertion [3]. It is used in practice for example in triangulation packages of CGAL [9]. This algorithm has been extended to periodic point sets, which can be seen as the image of a finite point set under the action of a group of translations [6, 5]. For example, given a finite point set in the unit square in the Euclidean plane and the group generated by the Euclidean translations of unit length in the x- and y-direction, one obtains a periodic point set in the Euclidean plane. Equivalently, this can be seen as a finite point set on the flat torus, where the flat torus is identified with the quotient space of the Euclidean plane under the action of the group mentioned above. If we consider the Delaunay triangulation of the periodic point set in the Euclidean plane, and project this triangulation to the flat torus, the result may be non-simplicial. Namely, if for example both endpoints of an edge project to the same point, then we obtain a loop. Since the incremental algorithm that we use assumes that the triangulation is a simplicial complex, this situation must be avoided. It is known that $1-$ and $2-$cycles can be avoided when the inequality $\text{sys}(\mathbb{M}) > 2\delta_{\mathcal{P}}$ is satisfied, where $\text{sys}(\mathbb{M})$ denotes the systole of the surface $\mathbb{M}$, i.e. the length of the shortest non-contractible curve, and $\delta_{\mathcal{P}}$ the diameter of a largest disk not containing any points from the input set $\mathcal{P}$ in its interior. Intuitively, this condition means that the length of every edge in the triangulation is less than $\frac{1}{2}\,\text{sys}(\mathbb{M})$, which implies that there can be no $1-$ or $2-$cycles. To make sure that this condition is satisfied, the triangulation can be initialized with a dummy point set, for which the inequality is satisfied by construction. After inserting the input points, the dummy points are removed (if possible).

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In this work, we will consider Delaunay triangulations on hyperbolic surfaces. Unlike Euclidean surfaces, in general the systole of a hyperbolic surface is unknown. An upper bound of order $O(\log g)$ is known [4, Lemma 5.2.1], where g denotes the genus, but lower bounds exist only for specific families of surfaces, often constructed using algebraic methods; in general the systole can be made arbitrarily small. The exact value of the systole is known for only a few specific hyperbolic surfaces. The Bolza surface, the most symmetric hyperbolic surface of genus 2, is one of the hyperbolic surfaces for which the systole is known. Here the regular hyperbolic octagon is a fundamental region for the group of translations. A generalization and implementation of the incremental algorithm for the Bolza surface is known [2, 8].

We will generalize the incremental algorithm for the Bolza surface to what we call 'symmetric hyperbolic surfaces' of arbitrary genus. Just like the Bolza surface (genus $g = 2$) corresponds to the regular hyperbolic octagon, the symmetric hyperbolic surface $\mathbb{M}_g$ of genus $g \geq 2$ corresponds to the regular hyperbolic $4g$-gon. Firstly, we derive the value of the systole of these surfaces to be able to verify the condition $\text{sys}(\mathbb{M}_g) > 2\delta_{\mathcal{P}}$. Secondly, we show that it is sufficient to consider a finite subset of the (infinite) periodic point set. Finally, we will construct for each symmetric hyperbolic surface a dummy point set with which to initialize the algorithm.

2 Preliminaries

2.1 Hyperbolic geometry

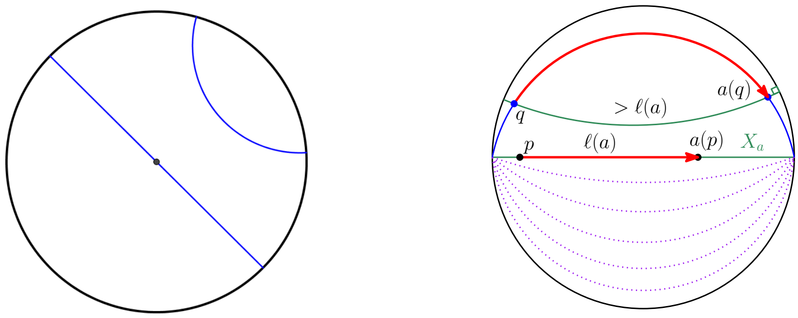
We will study periodic triangulations in the hyperbolic plane, or equivalently, triangulations on hyperbolic surfaces. There are several models for the hyperbolic plane; we will use the Poincaré disk model [1]. Here, the hyperbolic plane $\mathbb{H}^2$ is represented by the open unit disk $\mathbb{D}$ in the complex plane. This space is endowed with a specific metric, for which the hyperbolic lines (i.e. geodesics) are represented by diameters of $\mathbb{D}$ or circle arcs intersecting the boundary of $\mathbb{D}$ orthogonally (see Figure 1a). Isometries of $\mathbb{H}^2$ can be written in the form

$$f(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}},$$

where $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$. There are several types of isometries of $\mathbb{H}^2$, of which we will only consider *hyperbolic* isometries, also called translations, for which the real part of α is larger than 1. Every hyperbolic translation f has an invariant hyperbolic line, called the axis of f (see Figure 1b). A striking difference between translations of the Euclidean plane and translations of the hyperbolic plane is that the latter are in general not commutative.

▶ **Definition 2.1.** *A hyperbolic surface is a connected 2-dimensional manifold that is locally isometric to an open subset of* $\mathbb{H}^2$.

It is known that every hyperbolic surface $\mathbb{M}$ can be written as a quotient space $\mathbb{M} = \mathbb{H}^2/\Gamma$ of the hyperbolic plane under the action of a Fuchsian group Γ, i.e., a discrete subgroup of the group of isometries of $\mathbb{H}^2$ [11]. This quotient space can be represented by a polygon that is a fundamental region for the action of Γ, combined with a side pairing. In the case of the flat torus, the unit square is a fundamental polygon and the side pairings are given by the translations in the $x-$ and $y-$direction. For the Bolza surface, a fundamental domain is given by the regular hyperbolic octagon with total interior angle 2π, where opposite sides are paired (see Figure 2). In the same way, the symmetric hyperbolic surface $\mathbb{M}_g = \mathbb{H}^2/\Gamma_g$



(a) The Poincaré disk model of the hyperbolic plane with some hyperbolic lines

(b) A hyperbolic translation a, its axis X_a and the image of a point q not on the axis

Figure 1 Hyperbolic geometry

of genus $g \geq 2$ corresponds to the regular hyperbolic polygon F_g with total interior angle 2π, where opposite sides are paired.

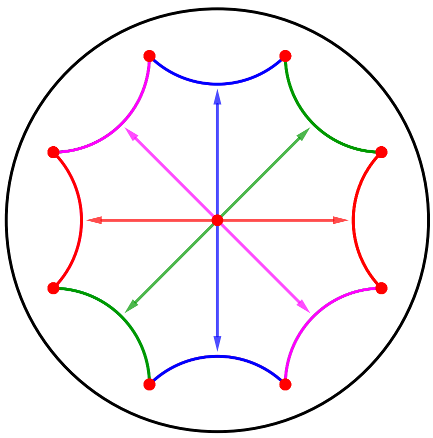


Figure 2 The Bolza surface

The action of the group Γ_g on the fundamental region F_g induces a tesselation of the hyperbolic plane into $4g$-gons. Because the interior angles of the $4g$-gon F_g add up to 2π, it can be seen that $4g$ polygons meet in every vertex in the tesselation. As a comparison, the group of translations of the flat torus induce a tesselation of the Euclidean plane into unit squares.

Finally, the systole of a hyperbolic surface $\mathbb{M}$ is the length of smallest non-contractible closed curve and is denoted by $\text{sys}(\mathbb{M})$. We will sketch a derivation of the value of the systole of the symmetric hyperbolic surfaces in Section 3.

2.2 Delaunay triangulations

In this work we will consider simplicial Delaunay triangulations, which satisfy the following two conditions:

- they are a simplicial complex,

- they satisfy the empty circle property.

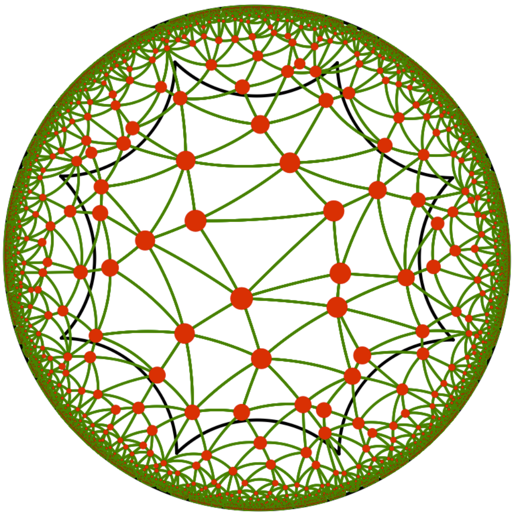


Figure 3 Example of a Delaunay triangulation of a periodic point set in the hyperbolic plane

Consider a finite point set $\mathcal{P}$ on a hyperbolic surface $\mathbb{M} = \mathbb{H}^2/\Gamma$. To define the Delaunay triangulation $\mathrm{DT}_{\mathbb{M}}(\mathcal{P})$ of $\mathcal{P}$ on $\mathbb{M}$, we can consider the images $\Gamma_g\mathcal{P}$ of $\mathcal{P}$ under the group action of Γ_g. Then we project the Delaunay triangulation $\mathrm{DT}_{\mathbb{H}}(\Gamma\mathcal{P})$ of the infinite point set $\Gamma\mathcal{P}$ in $\mathbb{H}^2$ to $\mathbb{M}$ using the universal covering map $\pi : \mathbb{H}^2 \to \mathbb{H}^2/\Gamma$. In that case, we could define "$\mathrm{DT}_{\mathbb{M}}(\mathcal{P}) = \pi(\mathrm{DT}_{\mathbb{H}}(\Gamma\mathcal{P}))$". However, the result is not necessarily a simplicial complex. It is known that the following criterion is sufficient to guarantee that $\pi(\mathrm{DT}_{\mathbb{H}}(\Gamma\mathcal{P}))$ is a simplicial complex [2]. Here $\delta_{\mathcal{P}}$ denotes the diameter of the largest disk in $\mathbb{H}^2$ that does not contain any point of $\Gamma\mathcal{P}$ in its interior.

▶ **Proposition 2.2.** *Let $\mathbb{M} = \mathbb{H}^2/\Gamma$ be a hyperbolic surface and $\mathcal{P} \subset \mathbb{M}$ a finite point set. If $\mathrm{sys}(\mathbb{M}) > 2\delta_{\mathcal{P}}$, then $\pi(\mathrm{DT}_{\mathbb{H}}(\Gamma\mathcal{P}))$ is a simplicial complex.*

Because $\delta_{\mathcal{Q}} \leq \delta_{\mathcal{P}}$ for $\mathcal{Q} \supseteq \mathcal{P}$, it follows that if $\mathrm{sys}(\mathbb{M}) > 2\delta_{\mathcal{P}}$ for some set $\mathcal{P}$, then the Delaunay triangulation of any superset of $\mathcal{P}$ is a simplicial complex as well. This makes the incremental algorithm work in this case.

3 Systole of symmetric hyperbolic surfaces

Recall that $\mathbb{M}_g$ denotes the symmetric hyperbolic surface of genus g. As mentioned before, to be able to verify the inequality $\mathrm{sys}(\mathbb{M}_g) > 2\delta_{\mathcal{P}}$, we have to know the value of $\mathrm{sys}(\mathbb{M}_g)$. This value is given in the following theorem.

▶ **Theorem 3.1.** *The systole of the surface $\mathbb{M}_g$ corresponding to the regular $4g$-gon satisfies*

$$\cosh\left(\tfrac{1}{2}\,\mathrm{sys}(\mathbb{M}_g)\right) = 1 + 2\cos\left(\tfrac{\pi}{2g}\right).$$

In the proof, we first show that there exists a closed, non-contractible curve with the stated length. To prove that there are no shorter such curves, we represent each closed geodesic on the surface $\mathbb{M}_g$ as a sequence of hyperbolic line segments between sides of the regular $4g$-gon F_g (see Figure 4) and analyze the different configurations of sequences of segments.

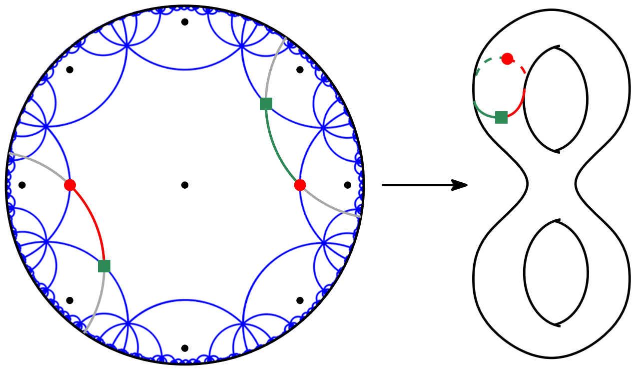


Figure 4 Representation of a systole of $\mathbb{M}_g$ as a sequence of segments between sides

4 Representation of Delaunay triangulations

In Section 2.2 we considered the Delaunay triangulation of $\Gamma_g\mathcal{P}$. However, practically speaking it is not possible to work with triangulations of point sets with infinitely many points. For this reason, let $D_\mathcal{N}$ be the union of translates of F_g, that share are least one point with F_g (see Figure 5).

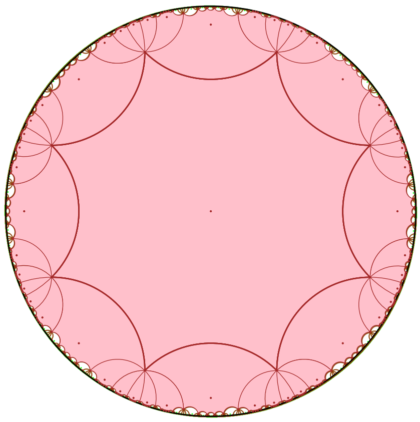


Figure 5 The union $D_\mathcal{N}$ of neighboring regions for the Bolza surface

Now, the next proposition states that it is sufficient to consider the combinatorics of the Delaunay triangulation of the points inside $D_\mathcal{N}$ if the inequality $\mathrm{sys}(\mathbb{M}_g) > 2\delta_\mathcal{P}$ is satisfied.

▶ **Proposition 4.1.** *Assume that $\mathcal{P}$ satisfies $\mathrm{sys}(\mathbb{M}_g) > 2\delta_\mathcal{P}$. Let Δ be a triangle in $\mathrm{DT}_\mathbb{H}(\Gamma_g\mathcal{P})$. If $\Delta \cap F_g \neq \emptyset$, then $\Delta \subset D_\mathcal{N}$.*

In other words, as soon as the condition $\mathrm{sys}(\mathbb{M}_g) > 2\delta_\mathcal{P}$ is satisfied, it suffices to compute the Delaunay triangulation of the finite point set $\Gamma_g\mathcal{P} \cap D_\mathcal{N}$ instead of the Delaunay triangulation of $\Gamma_g\mathcal{P}$. As we mentioned before, a dummy point set $\mathcal{Q}$ is used to guarantee that the condition holds. However, we cannot use Proposition 4.1 to find the Delaunay triangulation of the dummy point set. Instead we will use the following proposition. Here, $\mathcal{Q}_0$ denotes the set consisting of the origin, the vertex and the midpoints of the sides of F_g.

▶ **Proposition 4.2.** *Assume that $\mathcal{Q} \supseteq \mathcal{Q}_0$. Let Δ be a triangle in $\mathrm{DT}_\mathbb{H}(\Gamma_g\mathcal{Q})$. If $\Delta \cap F_g \neq \emptyset$, then $\Delta \subset D_\mathcal{N}$.*

Now, if we construct the dummy point set $\mathcal{Q}$ in such a way that it contains $\mathcal{Q}_0$, then we can find the Delaunay triangulation of $\mathcal{Q}$ by considering the Delaunay triangulation of $\Gamma_g\mathcal{Q}\cap D_{\mathcal{N}}$.

5 Initialization

In this section we will present an algorithm to compute a dummy point set, i.e., a finite point set $\mathcal{Q}$ that satisfies $\text{sys}(\mathbb{M}_g) > 2\delta_{\mathcal{Q}}$. The idea of this algorithm is similar to Delaunay refinement, also known as Ruppert's algorithm [10]. This works as follows. Initially, the dummy point set $\mathcal{Q}$ will only contain $\mathcal{Q}_0$. Then we consider the Delaunay triangulation $\text{DT}_{\mathbb{H}}(\Gamma_g\mathcal{Q}\cap D_{\mathcal{N}})$ of $\Gamma_g\mathcal{Q}\cap D_{\mathcal{N}}$ in $\mathbb{H}^2$. If there is a triangle in this triangulation with circumradius at least $\frac{1}{2}\,\text{sys}(\mathbb{M}_g)$, which has a non-empty intersection with the fundamental polygon, then the circumcenter of this triangle is added to $\mathcal{Q}$. This process continues until there are no more such triangles. See Figure 6 for an application of the algorithm to the symmetric hyperbolic surface of genus 3. A more formal description can be found below.

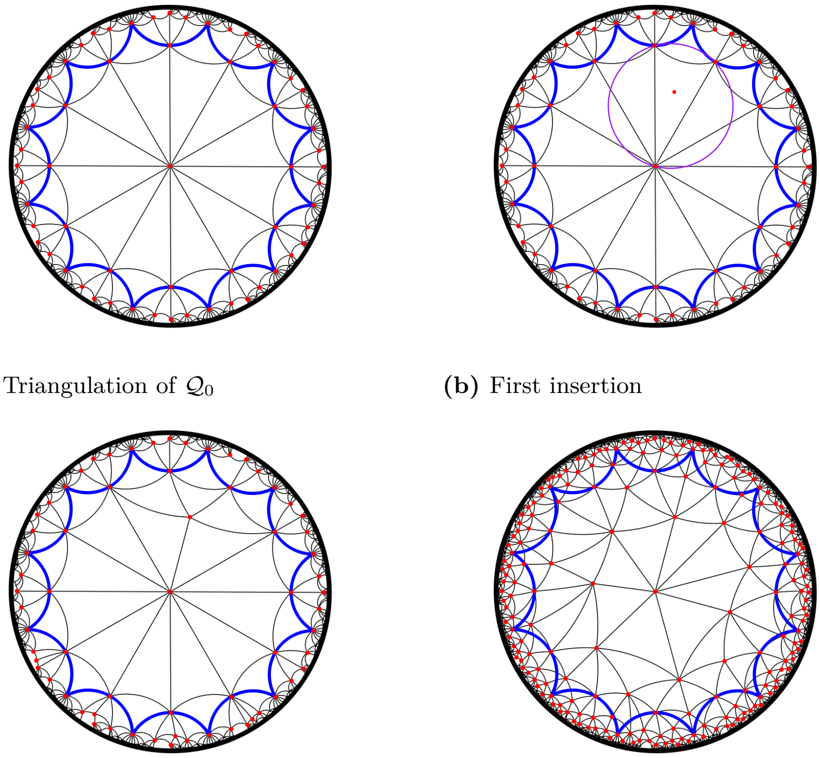
Algorithm 1: Dummy point algorithm

Input : hyperbolic surface $\mathbb{M}_g$
Output: finite point set $\mathcal{Q}\subset\mathbb{M}_g$ such that $\text{sys}(\mathbb{M}_g) > 2\delta_{\mathcal{Q}}$

```
Initialize: let Q = Q_0.
Compute DT_H(Γ_g Q ∩ D_N).
while there exists a triangle Δ in DT_H(Γ_g Q ∩ D_N) with circumdiameter at least
  1/2 sys(M_g) and Δ ∩ F_g ≠ ∅ do
    Add the circumcenter of Δ to Q
    Update DT_H(Γ_g Q ∩ D_N)
end
```

▶ **Theorem 5.1.** *The dummy point algorithm terminates. The resulting dummy point set $\mathcal{Q}$ satisfies $\text{sys}(\mathbb{M}_g) > 2\delta_{\mathcal{Q}}$ and has cardinality of order $\Theta(g)$.*

To give an idea of the proof that the number of iterations of the **while** loop is of order $\Theta(g)$ (and hence, that the algorithm terminates), we observe that the distance between every pair of points in $\mathcal{Q}$ is at least $\frac{1}{4}\,\text{sys}(\mathbb{M}_g)$. It follows that we can construct a circle packing on $\mathbb{M}_g$ with circles of radius $\frac{1}{8}\,\text{sys}(\mathbb{M}_g)$ centered at the points in $\mathcal{Q}$. Since the area of $\mathbb{M}_g$ is of order $\Theta(g)$, the above claims follows. Finally, we note that the size complexity of the resulting dummy point set is asymptotically optimal [7, Prop. 9.1].



(a) Triangulation of $\mathcal{Q}_0$

(b) First insertion

(c) After first insertion

(d) After last insertion

Figure 6 Several steps in the dummy point algorithm

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