Hamiltonicity for convex shape Delaunay and Gabriel graphs*

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Abstract

We study Hamiltonicity for some of the most general variants of Delaunay and Gabriel graphs. Let S be a point set in the plane. The k-order Delaunay graph of S, denoted k-$DG_{\mathcal{C}}(S)$, has vertex set S and edge pq provided that there exists *some* homothet of $\mathcal{C}$ with p and q on its boundary and containing at most k points of S different from p and q. The k-order Gabriel graph k-$GG_{\mathcal{C}}(S)$ is defined analogously, except for the fact that the homothets considered are restricted to be *smallest* homothets of $\mathcal{C}$ with p and q on its boundary. We provide upper bounds on the minimum value of k for which k-$GG_{\mathcal{C}}(S)$ is Hamiltonian. Since k-$GG_{\mathcal{C}}(S) \subseteq k$-$DG_{\mathcal{C}}(S)$, all results carry over to k-$DG_{\mathcal{C}}(S)$. In particular, we give upper bounds of 24 for every $\mathcal{C}$ and 15 for every point-symmetric $\mathcal{C}$. We also improve the bound to 7 for squares, 11 for regular hexagons, 12 for regular octagons, and 11 for even-sided regular t-gons (for $t \geq 10$).

1 Introduction

The study of the combinatorial properties of geometric graphs has played an important role in the area of Discrete and Computational Geometry. One of the fundamental structures that has been studied intensely is the *Delaunay triangulation* of a planar point set (see [9] for an encyclopedic treatment of this structure). It was conjectured by Shamos [10] that the Delaunay triangulation contains a Hamiltonian cycle. This was disproved by Dillencourt [5]. However, Dillencourt [6] showed that Delaunay triangulations are *almost* Hamiltonian, in the sense that they are 1-tough.[1]

Focus then shifted on determining how much the definition of the Delaunay triangulation can be loosened to achieve Hamiltonicity. To this end, Chang et al. [4] showed that the 19-Delaunay graph is Hamiltonian.[2] Given a planar point set S, the *k-Delaunay graph* has vertex set S and edge pq provided that there exists a disk with p and q on the boundary

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[1] A graph is *1-tough* if removing any k vertices from it results in at most k connected components.
[2] According to the definition of k-DG in [4], they showed Hamiltonicity for 20-DG.

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Type of shape $\mathcal{C}$	k
Convex	24
Point-symmetric convex	15
Regular octagons	12
Regular hexagons & even-sided regular t-gons, with $t \geq 10$	11
Squares	7

Table 1 Obtained upper bounds on the minimum k for which k-$GG_{\mathcal{C}}(S)$ is Hamiltonian.

containing at most k points of S different from p and q.[3] If the disk with p and q on its boundary is restricted to disks with pq as diameter, then the graph is called the k-Gabriel graph and is a subgraph of the k-Delaunay graph. In fact, Chang et al. [4] showed that the 19-Gabriel graph is Hamiltonian. This was subsequently lowered to $k = 15$ [1] and the current best bound is $k = 10$ [8]. It is conjectured that 1-Delaunay is Hamiltonian [1].

In this article, we generalize the above results by replacing the disk with an arbitrary convex shape. We show that the k-Gabriel graph is Hamiltonian for any convex shape $\mathcal{C}$ when $k \geq 24$, and give improved bounds for various more specific convex shapes. Table 1 summarizes the bounds obtained. Our results rely on tools from metrics and packings.

2 Convex distances and the $\mathcal{C}$-Gabriel graph

Let p and q be two points in the plane. Let $\mathcal{C}$ be a compact convex set that contains the origin, denoted $\bar{o}$, in its interior. The *convex distance* $d_{\mathcal{C}}(p, q)$ is defined in the following way: If $p = q$, then $d_{\mathcal{C}}(p, q) = 0$. Otherwise, $d_{\mathcal{C}}(p, q) = \frac{d(p,q)}{d(p,q')}$, where q' is the intersection of the ray from p to q with the translate of $\mathcal{C}$ by the vector $\overrightarrow{\bar{o}p}$ (see Figure 1). The convex set $\mathcal{C}$ is the *unit $\mathcal{C}$-disk* of $d_{\mathcal{C}}$ with center $\bar{o}$, i.e., every point p in $\mathcal{C}$ satisfies that $d_{\mathcal{C}}(\bar{o}, p) \leq 1$. The *$\mathcal{C}$-disk with center c and radius r* is defined as the homothet of $\mathcal{C}$ centered at c and with scaling factor r.

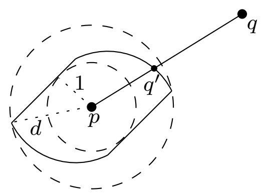

Figure 1 Convex distance from p to q.

The *triangle inequality* holds: $d_{\mathcal{C}}(p, q) \leq d_{\mathcal{C}}(p, r) + d_{\mathcal{C}}(r, q), \forall p, q, r \in \mathbb{R}^2$. However, this distance may not define a metric when $\mathcal{C}$ is not *point-symmetric*[4] about the origin, since there may be points p, q for which $d_{\mathcal{C}}(p, q) \neq d_{\mathcal{C}}(q, p)$. When $\mathcal{C}$ is point-symmetric with respect to the origin, $d_{\mathcal{C}}$ is called a *symmetric convex distance function*. Such a distance defines a metric; moreover, $d_{\mathcal{C}}(\bar{o}, p)$ defines a *norm*[5] of a *metric space*. In addition, if a point p is on the line segment ab, then $d_{\mathcal{C}}(a, b) = d_{\mathcal{C}}(a, p) + d_{\mathcal{C}}(p, b)$ (see [2, Chapter 7]).

[3] Note that this implies that the standard Delaunay triangulation is the 0-Delaunay graph.

[4] A shape $\mathcal{C}$ is point-symmetric with respect to a point $x \in \mathcal{C}$ provided that for every point $p \in \mathcal{C}$ there is a corresponding point $q \in \mathcal{C}$ such that $pq \in \mathcal{C}$ and x is the midpoint of pq.

[5] A function $\rho(x)$ is a norm if: (a) $\rho(x) = 0$ if and only if $x = \bar{o}$, (b) $\rho(\lambda x) = |\lambda| \rho(x)$ where $\lambda \in \mathbb{R}$, and (c) $\rho(x + y) \leq \rho(x) + \rho(y)$.

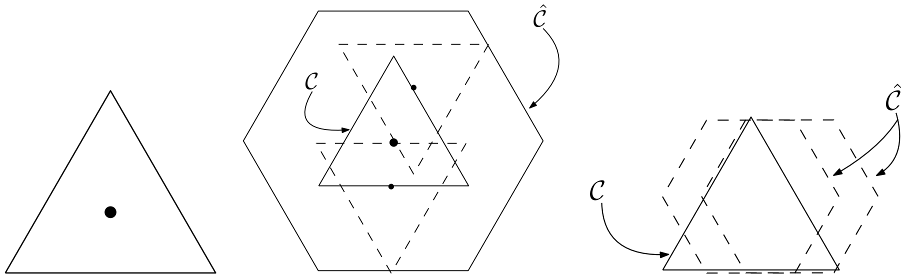

Figure 2 Left: A triangle is a $\mathcal{C}$ shape. Center: $\hat{\mathcal{C}}$ for this triangle is a hexagon. Right: the shape $\hat{\mathcal{C}}$ with radius $\frac{1}{2}$ does not contain $\mathcal{C}$.

Let S be a set of points in the plane satisfying the following general position assumption: For each pair $p, q \in S$, any minimum homothet of $\mathcal{C}$ having p and q on its boundary does not contain any other point of S on its boundary. The *k-order $\mathcal{C}$-Delaunay graph* of S, denoted k-$DG_{\mathcal{C}}(S)$, is the graph with vertex set S such that, for each pair of points $p, q \in S$, the edge pq is in k-$DG_{\mathcal{C}}(S)$ if there exists a $\mathcal{C}$-disk that has p and q on its boundary and contains at most k points of S different from p and q. When $k = 0$ and $\mathcal{C}$ is a circle, k-$DG_{\mathcal{C}}(S)$ is the standard *Delaunay triangulation*.

Aurenhammer and Paulini [3] showed how to define a point-symmetric distance function from any convex shape $\mathcal{C}$, as follows. Denote by $\mathcal{C}_v$ the shape $\mathcal{C}$ with $\bar{o}$ translated by vector v. The distance from p to q is given by the scaling factor of a smallest homothet containing p and q on its boundary, which is equivalent to $\min_{v \in \mathcal{C}} d_{\mathcal{C}_v}(p, q) = d_{\hat{\mathcal{C}}}(p, q)$ where $\hat{\mathcal{C}} = \bigcup_{v \in \mathcal{C}} \mathcal{C}_v$. The set $\hat{\mathcal{C}}$ is a point-symmetric convex set that is the Minkowski sum[6] of $\mathcal{C}$ and its shape reflected about its center. For an example, see Figure 2. The diameter and width of $\hat{\mathcal{C}}$ is twice the diameter and width of $\mathcal{C}$, respectively. Moreover, when $\mathcal{C}$ is point-symmetric, $d_{\hat{\mathcal{C}}}(p, q) = \frac{d_{\mathcal{C}}(p,q)}{2}$.

We define the *k-order $\mathcal{C}$-Gabriel graph* of S, denoted k-$GG_{\mathcal{C}}(S)$, as the graph with vertex set S such that, for every pair of points $p, q \in S$, the edge pq is in k-$GG_{\mathcal{C}}(S)$ if and only if there exists a $\mathcal{C}$-disk with radius $d_{\hat{\mathcal{C}}}(p, q)$ that has p and q on its boundary and contains at most k points of S different from p and q. From the definition of k-$GG_{\mathcal{C}}(S)$ and k-$DG_{\mathcal{C}}(S)$ we note that k-$GG_{\mathcal{C}}(S) \subseteq k$-$DG_{\mathcal{C}}(S)$, and it can be a proper subgraph. See Figure 3 for an example. Further, when $\mathcal{C}$ is not point-symmetric, then $\hat{\mathcal{C}}$ contains $\mathcal{C}$ in its interior; however, for some shapes it is not true that the $\hat{\mathcal{C}}$-disk with radius $\frac{1}{2}$ contains $\mathcal{C}$ (refer to Figure 2, right). Thus, for asymmetric shapes $\mathcal{C}$, in general $GG_{\hat{\mathcal{C}}} \not\subseteq GG_{\mathcal{C}}$.

3 Hamiltonicity for convex shapes

3.1 General convex shapes

Define $\mathcal{H}$ to be the set of all Hamiltoninan cycles of the point set S. Define the $d_{\hat{\mathcal{C}}}$-length sequence of $h \in \mathcal{H}$, denoted $ds_{\mathcal{C}}(h)$, as the edge sequence sorted in decreasing order with respect to the length of the edges in $d_{\hat{\mathcal{C}}}$-metric. Sort the elements of $\mathcal{H}$ in lexicographic order with respect to their $d_{\hat{\mathcal{C}}}$-length sequence, breaking ties arbitrarily. This order is strict. For $h_1, h_2 \in \mathcal{H}$, if h_1 is smaller than h_2 in this order, we write $h_1 \prec h_2$.

[6] The Minkowski sum of two sets A and B is defined as $A \oplus B = \{a + b : a \in A, b \in B\}$.

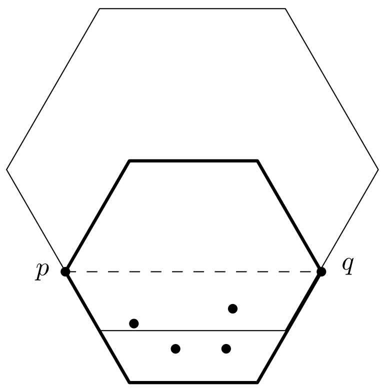

Figure 3 $\mathcal{C}$ is a regular hexagon. Edge pq is in 2-$DG_{\mathcal{C}}(S)$ but it is not in 2-$GG_{\mathcal{C}}(S)$.

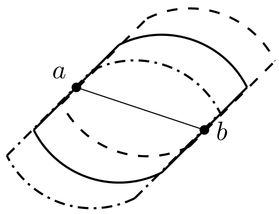

Figure 4 Many $\mathcal{C}$-disks $\mathcal{C}(a,b)$ may exist for a and b.

For simplicity, denote by $\mathcal{C}_r(a,b)$, a $\mathcal{C}$-disk with radius r containing the points a and b on its boundary. For the special case of a *diametral disk*, i.e., when the radius of $\mathcal{C}_r(a,b)$ is $d_{\hat{\mathcal{C}}}(a,b)$, we denote it as $\mathcal{C}(a,b)$. Note that $\mathcal{C}(a,b)$ may not be unique, see Figure 4. In addition, we denote by $D_{\mathcal{C}}(c,r)$ the $\mathcal{C}$-disk centered at point c with radius r.

▶ Claim 3.1. Let $\mathcal{C}$ be a point-symmetric convex shape. Let u be a point in the plane different from the origin $\bar{o}$. Let $r < d_{\mathcal{C}}(u,\bar{o})$. Let p be the intersection point of $D_{\mathcal{C}}(u,r)$ and line segment $\bar{o}u$. Let $u' = \lambda u$, with $\lambda > 1 \in \mathbb{R}$, be a point defined by vector u scaled by a factor of λ. Then $D_{\mathcal{C}}(u,r) \subset D_{\mathcal{C}}(u', d_{\mathcal{C}}(u',p))$. (See Figure 5.)

Proof. Let $q \in D_{\mathcal{C}}(u,r)$; then $d_{\mathcal{C}}(u,q) \leq d_{\mathcal{C}}(u,p)$. Since u is on the line segment $u'p$, we have that $d_{\mathcal{C}}(u',p) = d_{\mathcal{C}}(u',u) + d_{\mathcal{C}}(u,p)$. Hence $d_{\mathcal{C}}(u',q) \leq d_{\mathcal{C}}(u',u) + d_{\mathcal{C}}(u,q) \leq d_{\mathcal{C}}(u',u) + d_{\mathcal{C}}(u,p) = d_{\mathcal{C}}(u',p)$. Therefore, $D_{\mathcal{C}}(u,r)$ is contained in $D_{\mathcal{C}}(u', d_{\mathcal{C}}(u',p))$. ◀

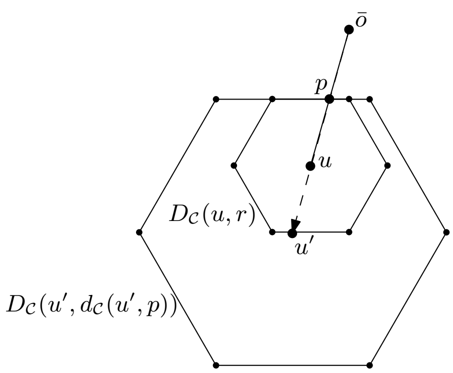

Figure 5 $D_{\mathcal{C}}(u,r)$ is contained in $D_{\mathcal{C}}(u', d_{\mathcal{C}}(u',p))$, where $u' = \lambda u$ with $\lambda > 1$.

The approach we follow to prove our bounds, which is similar to the approach in [1, 4, 8], is to show that the minimum element in $\mathcal{H}$ is contained in k-$GG_{\mathcal{C}}(S)$ for a small value of k. Let h be the minimum element in $\mathcal{H}$. Let $ab \in h$; we can assume without loss of generality that $d_{\hat{\mathcal{C}}}(a,b) = 1$. Let $U = \{u_1, u_2, \dots, u_k\}$ be the set of points in S different from a and b that are in the interior of $\mathcal{C}(a,b)$.[7] We assume that, when traversing h from b to a, we visit the points of U in order $u_1, \dots, u_k$. For each point u_i, we define s_i to be the point preceding u_i in h. See Figure 6.

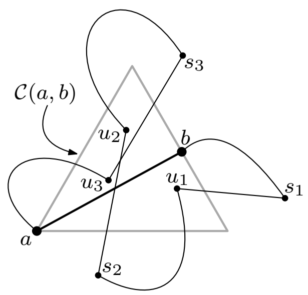

Figure 6 Example of U in $\mathcal{C}(a,b)$.

Note that if a point p is in the interior of $\mathcal{C}(a,b)$, then there exists a $\mathcal{C}(p,q)$ contained in $\mathcal{C}(a,b)$ for any point q on the boundary of $\mathcal{C}(a,b)$. Therefore, $d_{\hat{\mathcal{C}}}(a,u_i) < 1$ and $d_{\hat{\mathcal{C}}}(b,u_i) < 1$ for any $i \in \{1,\dots,k\}$. Furthermore, we have the following:

▶ Claim 3.2. Let $1 \le i \le k$. Then $d_{\hat{\mathcal{C}}}(a,s_i) \ge \max\{d_{\hat{\mathcal{C}}}(s_i,u_i), 1\}$.

Proof. If $s_1 = b$, then $d_{\hat{\mathcal{C}}}(a,s_1) = 1$ and $d_{\hat{\mathcal{C}}}(s_1,u_1) < 1$. Otherwise, we define $h'=(h \setminus \{ab, s_iu_i\}) \cup \{as_i, u_ib\}$. For the sake of a contradiction, assume that $d_{\hat{\mathcal{C}}}(a,s_i) < \max\{d_{\hat{\mathcal{C}}}(s_i,u_i), 1\}$. Since $d_{\hat{\mathcal{C}}}(a,b) = 1$, this implies that $d_{\hat{\mathcal{C}}}(a,s_i) < \max\{d_{\hat{\mathcal{C}}}(s_i,u_i), d_{\hat{\mathcal{C}}}(a,b)\}$. Moreover, since $u_i \in \mathcal{C}(a,b)$, we have $d_{\hat{\mathcal{C}}}(u_i,b) < 1$. Thus, $\max\{d_{\hat{\mathcal{C}}}(a,s_i), d_{\hat{\mathcal{C}}}(u_i,b)\} < \max\{d_{\hat{\mathcal{C}}}(s_i,u_i), d_{\hat{\mathcal{C}}}(a,b)\}$. Therefore $h' \prec h$, which contradicts the definition of h. ◀

Claim 3.2 implies that, for each $i \in \{1,\dots,k\}$, the point s_i is not in the interior of $\mathcal{C}(a,b)$.

▶ Claim 3.3. Let $1 \le i < j \le k$. Then $d_{\hat{\mathcal{C}}}(s_i,s_j) \ge \max\{d_{\hat{\mathcal{C}}}(s_i,u_i), d_{\hat{\mathcal{C}}}(s_j,u_j), 1\}$.

The proof of this claim is similar to the proof of Claim 3.2.

Without loss of generality we assume that a is the origin $\bar{o}$. Then, by the definition of $\hat{\mathcal{C}}$, we have that $D_{\hat{\mathcal{C}}}(\bar{o},1)$ contains $\mathcal{C}(a,b)$. Also, from Claim 3.2, we have that s_i is not in the interior of $D_{\hat{\mathcal{C}}}(\bar{o},1)$ for all $i \in \{1,\dots,k\}$. Let $D_{\hat{\mathcal{C}}}(\bar{o},2)$ be the $\hat{\mathcal{C}}$-disk centered at a with radius 2. For each $s_i \notin D_{\hat{\mathcal{C}}}(\bar{o},2)$, define s'_i as the intersection of $D_{\hat{\mathcal{C}}}(\bar{o},2)$ with the ray $\overrightarrow{as_i}$. We let $s'_i = s_i$ when s_i is inside $D_{\hat{\mathcal{C}}}(\bar{o},2)$. See Figure 7.

▶ Observation 3.4. If $s_j \notin D_{\hat{\mathcal{C}}}(\bar{o},2)$ (with $1 \le j \le k$), the $d_{\hat{\mathcal{C}}}$-distance from s'_j to $D_{\hat{\mathcal{C}}}(\bar{o},1)$ is 1.

▶ **Lemma 3.5.** *For any pair s_i and s_j with $i \neq j$, we have that $d_{\hat{\mathcal{C}}}(s'_i,s'_j) \ge 1$.*

Proof. If both s_i and s_j are in $D_{\hat{\mathcal{C}}}(\bar{o},2)$, then from Claim 3.3 we have that $d_{\hat{\mathcal{C}}}(s'_i,s'_j) = d_{\hat{\mathcal{C}}}(s_i,s_j) \ge 1$. In the following, we assume, without loss of generality, that $d_{\hat{\mathcal{C}}}(\bar{o},s_j) \ge d_{\hat{\mathcal{C}}}(\bar{o},s_i)$. Since s'_j is on the line segment $\bar{o}s_j$, we have $s_j = \lambda s'_j$ for some $\lambda > 1 \in \mathbb{R}$. Let p be the intersection point of $D_{\hat{\mathcal{C}}}(\bar{o},1)$ and $\bar{o}s_j$. Since $d_{\hat{\mathcal{C}}}$ defines a norm, we have

[7] By our general position assumption, the only points of S on the boundary of $\mathcal{C}(a,b)$ are a and b.

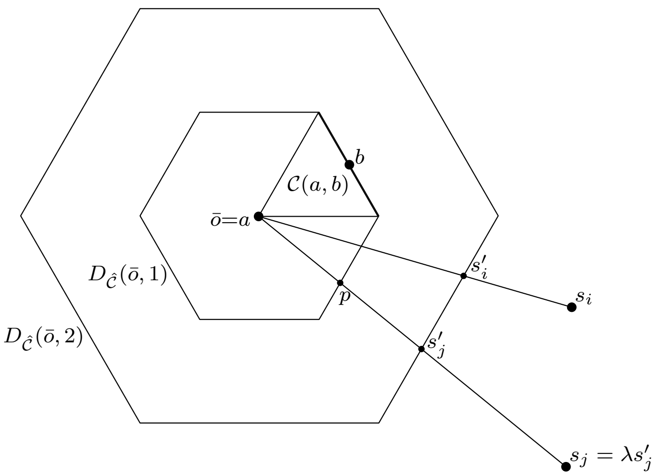

Figure 7 The points s'_i and s'_j are projections of s_i and s_j on $D_{\hat{\mathcal{C}}}(\bar{o},2)$, respectively.

$d_{\hat{\mathcal{C}}}(\lambda s'_j,\bar{o}) = \lambda d_{\hat{\mathcal{C}}}(s'_j,\bar{o})$. By Observation 3.4 we have that $d_{\hat{\mathcal{C}}}(s_j,p) = d_{\hat{\mathcal{C}}}(s_j,\bar{o}) - d_{\hat{\mathcal{C}}}(p,\bar{o}) = \lambda d_{\hat{\mathcal{C}}}(s'_j,\bar{o}) - 1 = 2\lambda - 1$, which is the distance from s_j to $D_{\hat{\mathcal{C}}}(\bar{o},1)$. Further, $d_{\hat{\mathcal{C}}}(s_j,s'_j) = d_{\hat{\mathcal{C}}}(s_j,\bar{o}) - d_{\hat{\mathcal{C}}}(s'_j,\bar{o}) = 2\lambda - 2$. For the sake of a contradiction, assume that $d_{\hat{\mathcal{C}}}(s'_i,s'_j) \leq 1$. If $s_j \notin D_{\hat{\mathcal{C}}}(\bar{o},2)$, we consider two cases:
Case 1) $s_i \in D_{\hat{\mathcal{C}}}(\bar{o},2)$. Then $d_{\hat{\mathcal{C}}}(\bar{o},s_i) \leq 2$. Let $D_{s'_j} = D_{\hat{\mathcal{C}}}(s'_j,1)$. Since $d_{\hat{\mathcal{C}}}(s'_i,s'_j) \leq 1$, we have $s_i \in D_{s'_j}$. From Claim 3.1 it follows that $d_{\hat{\mathcal{C}}}(s_j,s'_i) = d_{\hat{\mathcal{C}}}(s_j,s_i) \leq d_{\hat{\mathcal{C}}}(s_j,p) < d_{\hat{\mathcal{C}}}(s_j,u_j)$, which contradicts Claim 3.3.
Case 2) $s_i \notin D_{\hat{\mathcal{C}}}(\bar{o},2)$. Then $d_{\hat{\mathcal{C}}}(\bar{o},s_i) > 2$. Thus, $s_i = \delta s'_i$ for some $\delta > 1 \in \mathbb{R}$. Moreover, since $d_{\hat{\mathcal{C}}}(\bar{o},s_j) \geq d_{\hat{\mathcal{C}}}(\bar{o},s_i)$ and s'_i, s'_j are on the boundary of $D_{\hat{\mathcal{C}}}(\bar{o},2)$, $\delta \leq \lambda$. Hence, s_i is on the line segment $s'_i(\lambda s'_i)$. Let $D_{s_j} = D_{\hat{\mathcal{C}}}(s_j, 2\lambda - 1)$. Note that $\lambda < 2\lambda - 1$ because $\lambda > 1$. Since $d_{\hat{\mathcal{C}}}$ defines a norm, $d_{\hat{\mathcal{C}}}(s_j,\lambda s'_i) = d_{\hat{\mathcal{C}}}(\lambda s'_j,\lambda s'_i) = \lambda d_{\hat{\mathcal{C}}}(s'_j,s'_i) \leq \lambda < 2\lambda - 1$. Hence, $\lambda s'_i \in D_{s_j}$. In addition, from Claim 3.1 it follows that $D_{s'_j} \subseteq D_{s_j}$. Therefore, $s'_i \in D_{s_j}$. Thus, the line segment $s'_i(\lambda s'_i)$ is contained in D_{s_j}. Hence, $s_i \in D_{s_j}$. Then, $d_{\hat{\mathcal{C}}}(s_j,s_i) \leq 2\lambda - 1 = d_{\hat{\mathcal{C}}}(s_j,p) < d_{\hat{\mathcal{C}}}(s_j,u_j)$ which contradicts Claim 3.3. ◀

▶ **Theorem 3.6.** *For any set of points S in general position and convex shape $\mathcal{C}$, the graph 24-$GG_{\mathcal{C}}(S)$ is Hamiltonian.*

Proof. For each s_i we define the $\hat{\mathcal{C}}$-disk $D_i = D_{\hat{\mathcal{C}}}(s'_i, \frac{1}{2})$. We also set $D_0 := D_{\hat{\mathcal{C}}}(a, \frac{1}{2})$. By Lemma 3.5, each pair of $\hat{\mathcal{C}}$-disks D_i and D_j $(i \neq j)$ are internally disjoint. See Figure 8. Since $s'_i \in D_{\hat{\mathcal{C}}}(\bar{o},2)$ for all i, each disk D_i is inside $D_{\hat{\mathcal{C}}}(a, \frac{5}{2})$. There can be at most $\frac{Area(D_{\hat{\mathcal{C}}}(\bar{o},\frac{5}{2}))}{Area(D_0)} = \frac{(\frac{5}{2})^2 Area(\hat{\mathcal{C}})}{(\frac{1}{2})^2 Area(\hat{\mathcal{C}})} = 25$ disjoint disks in $D_{\hat{\mathcal{C}}}(\bar{o},2)$. Thus, there are at most 24 points s'_i in $D_{\hat{\mathcal{C}}}(\bar{o},1)$, since D_0 is centered at a. Hence, there are at most 24 points in the interior of $\mathcal{C}(a,b)$. ◀

3.2 Point-symmetric convex shapes

Using the fact that $d_{\mathcal{C}}$ defines a metric when $\mathcal{C}$ is point-symmetric, we can improve the upper bound for point-symmetric convex shapes. Indeed, given that $d_{\mathcal{C}} = 2d_{\hat{\mathcal{C}}}$ we can prove that: (i) $d_{\mathcal{C}}(s_i,a) \geq \max\{d_{\mathcal{C}}(s_i,u_i), 2\}$; and (ii) $d_{\mathcal{C}}(s_i,s_j) \geq \max\{d_{\mathcal{C}}(s_i,u_i), d_{\mathcal{C}}(s_j,u_j), 2\}$, for any $1 \leq i < j \leq k$. By using $\mathcal{C}(a,b)$ instead of $D_{\hat{\mathcal{C}}}(\bar{o},1)$, $D_{\mathcal{C}}(\bar{o},3)$ instead of $D_{\hat{\mathcal{C}}}(\bar{o},2)$, and $D_{\mathcal{C}}(\bar{o},4)$ instead of $D_{\hat{\mathcal{C}}}(\bar{o},\frac{5}{2})$, in combination with arguments similar to those in the previous section, we obtain that 15-$GG_{\mathcal{C}}$ is Hamiltonian.

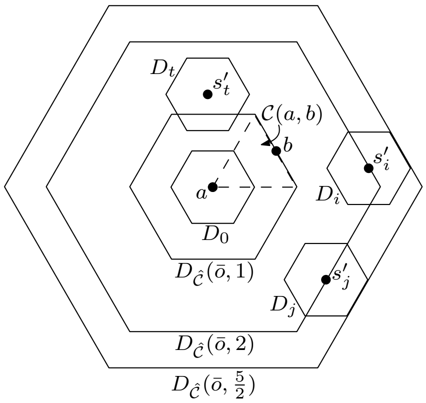

Figure 8 The $\hat{\mathcal{C}}$-disks D_i, D_j and D_t are contained in $D_{\hat{\mathcal{C}}}(a, \frac{5}{2})$.

When $\mathcal{C}$ is a regular polygon $\mathcal{P}_t$ with t even sides we can improve this bound by analyzing specific values of t. When $\mathcal{C}$ is a square, we divide $D_{\mathcal{P}_4}(\bar{o}, 3)$ into nine disjoint squares of radius 1 and show that only seven of them can contain points from $\{s'_1, \dots, s'_k\}$, with at most one point in each square. Using similar arguments as those for squares, we show that 11-$GG_{\mathcal{P}_6}$ is Hamiltonian. Finally, for the remaining regular polygons with even sides we use that the ex-circle of $D_{\mathcal{P}_{10}}(\bar{o}, 3)$ contains $D_{\mathcal{P}_t}(\bar{o}, 3)$ for all even $t \geq 10$. Such a circle has radius $r \approx 3.154$. Hence, we can prove Hamiltonicity for 11-$GG_{\mathcal{P}_t}$ using a result by Fodor [7] that states that the minimum radius of a circle having 13 points at pairwise Euclidean distance at least 2 is $R \approx 3.236$, which is greater than r. Analogously, we show that there are at most 13 points inside $D_{\mathcal{P}_8}(\bar{o}, 3)$ such that each pair is at Euclidean distance at least 2, which proves Hamiltonicity for 12-$GG_{\mathcal{P}_8}$.

References

1. Manuel Abellanas, Prosenjit Bose, Jesús García-López, Ferran Hurtado, Carlos M. Nicolás, and Pedro Ramos. On structural and graph theoretic properties of higher order Delaunay graphs. *Internat. J. Comput. Geom. Appl.*, 19(6):595–615, 2009.
2. Franz Aurenhammer, Rolf Klein, and Der-Tsai Lee. *Voronoi diagrams and Delaunay triangulations*. World Scientific Publishing Company, 2013.
3. Franz Aurenhammer and Günter Paulini. On shape Delaunay tessellations. *Inf. Process. Lett.*, 114(10):535–541, 2014.
4. Maw-Shang Chang, Chuan Yi Tang, and Richard C. T. Lee. 20-relative neighborhood graphs are Hamiltonian. *J. Graph Theory*, 15(5):543–557, 1991.
5. Michael B. Dillencourt. A non-Hamiltonian, nondegenerate Delaunay triangulation. *Inf. Process. Lett.*, 25(3):149–151, 1987.
6. Michael B. Dillencourt. Toughness and Delaunay triangulations. *Discrete Comput. Geom.*, 5:575–601, 1990.
7. Ferenc Fodor. The densest packing of 13 congruent circles in a circle. *Beitr. Algebra Geom.*, 44(2):431–440, 2003.
8. Tomáš Kaiser, Maria Saumell, and Nico Van Cleemput. 10-Gabriel graphs are Hamiltonian. *Inf. Process. Lett.*, 115(11):877–881, 2015.
9. Atsuyuki Okabe, Barry Boots, Kokichi Sugihara, and Sung Nok Chiu. *Spatial Tessellations: Concepts and Applications of Voronoi Diagrams*. Wiley, 2000.
10. Michael Shamos. Computational geometry. *PhD Thesis, Yale University*, 1978.