Unbounded Regions of Higher-Order Line and Segment Voronoi Diagrams in Higher Dimensions*

Gill Barequet[1], Evanthia Papadopoulou[2], and Martin Suderland[3]

1 **Dept. of Computer Science, The Technion—Israel Inst. of Technology, Haifa 3200003, Israel**
barequet@cs.technion.ac.il
2 **Faculty of Informatics, Università della Svizzera italiana, Lugano, Switzerland**
evanthia.papadopoulou@usi.ch
3 **Faculty of Informatics, Università della Svizzera italiana, Lugano, Switzerland**
martin.suderland@usi.ch

Abstract

We study the behavior at infinity of farthest and higher-order Voronoi diagrams of line segments or/and lines in a d-dimensional Euclidean space. The unbounded parts of these diagrams can be encoded by a *Gaussian map* on the sphere of directions $\mathbb{S}^{d-1}$. We show that the combinatorial complexity of the Gaussian map for the order-k Voronoi diagram of n line segments or/and lines is $O(\min(k, n-k)n^{d-1})$, which is tight for $n-k = O(1)$. The Gaussian map of the farthest Voronoi diagram of line segments or/and lines in $\mathbb{R}^3$ can be constructed in $O(n^2)$ time.

1 Introduction

The Voronoi diagram of a set of n geometric objects, called sites, is a well-known geometric space-partitioning structure. The *nearest* variant partitions the underlying space into maximal regions, such that all points within a region have the same nearest site. A very well-studied type of Voronoi diagram is the Euclidean Voronoi diagram of n points in $\mathbb{R}^d$, see [4, 7, 9].

Many algorithmic paradigms, such as plane sweep, incremental construction, and divide and conquer have been applied to construct the Voronoi diagram of line segments in the plane [2]. Already in a three-dimensional space, the algebraic description of the features, such as the edges, of the Voronoi diagram of line segments become complicated [8].

The order-k (resp., farthest) Voronoi diagram of a set of sites is a partition of the underlying space into regions, such that the points of one region have the same k nearest sites (resp., same farthest site). In two dimensions, the farthest Voronoi diagram of n segments has already been studied by Aurenhammer et al. [1], who give results on its structure and an algorithm to compute it in $O(n \log n)$ time. The order-k counterpart of this diagram has $O(k(n-k))$ complexity and it can be constructed iteratively [11]. Already in a three-dimensional space with the Euclidean metric, no tight asymptotic bound on the complexity of the farthest Voronoi diagram is known, and similarly for the nearest-neighbor diagram [10]. In both cases, the only known bounds are $\Omega(n^2)$ and $O(n^{3+\varepsilon})$, for any $\varepsilon > 0$ [3, 12].

The Euclidean farthest-neighbor Voronoi diagrams of lines and/or line segments in three dimensions has the property that all cells are unbounded [3]. This property motivated us to first study the unbounded parts of the farthest Voronoi diagram. Barequet and Papadopoulou [3] introduced a structure on the sphere of directions, called the Gaussian map, describing those unbounded parts. The Gaussian map associates with each cell of a diagram its unbounded directions. This results in a subdivision of the sphere of directions.

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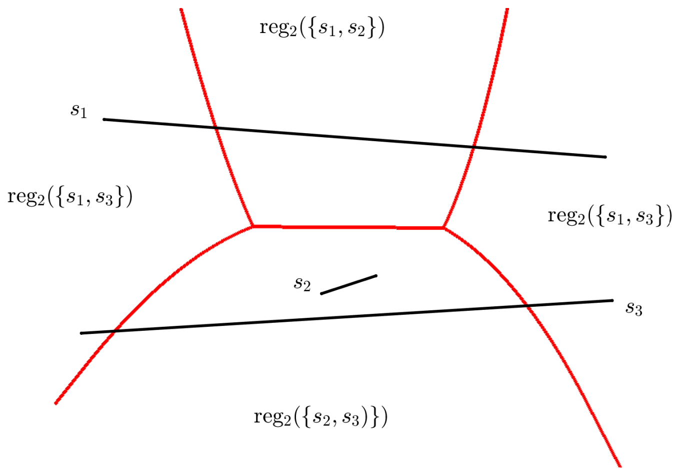

Figure 1 The order-2 Voronoi diagram (in red) of three segments s_1, s_2, s_3 in the plane.

In the current paper, we study the Gaussian map of order-k (and farthest) Voronoi diagrams of line segments and lines as sites in $\mathbb{R}^d$. We characterize the unbounded directions of the cells in these diagrams. We derive the bound $O(\min(k, n-k)n^{d-1})$ on the complexity of the Gaussian map of order-k Voronoi diagrams for these sites. We prove that when sites are segments, the complexity of the Gaussian map is $\Omega(k^{d-1})$, which is tight when $n - k = O(1)$. We also derive the bound $\Omega(k^{d-1})$ on the complexity of the entire order-k Voronoi diagram. We state an algorithm which computes the Gaussian map of the farthest Voronoi diagram in a three-dimensional space in worst-case optimal time $O(n^2)$.

2 Preliminaries

2.1 Order-k Voronoi diagrams

Let S be a set of non-intersecting sites in $\mathbb{R}^d$. In this paper, we consider n line segments or/and lines in $\mathbb{R}^d$ as sites. We assume that the directions of any d lines are linearly independent and no $d+1$ sites touch the same hyperplane, where sites contained in that hyperplane are counted twice. We denote by $d(x, y)$ the Euclidean distance between two points $x, y \in \mathbb{R}^d$. The distance $d(x, s)$ from a point $x \in \mathbb{R}^d$ to a site $s \in S$ is defined as $d(x, s) = min\{d(x, s) | y \in s\}$.

▶ **Definition 2.1.** For a subset of sites $H \subset S$ of cardinality $|H| = k$, the *order-k region* of H is the set of points in $\mathbb{R}^d$ whose distance to any site in H is smaller than to any site not in H, denoted as

$$\text{reg}_k(H) = \{p \in \mathbb{R}^d \mid \forall h \in H \ \forall s \in S \setminus H : \ d(p, h) < d(p, s)\}.$$

The order-k regions form a subdivision of $\mathbb{R}^d$. The induced cell complex, denoted by $\text{VD}_k(S)$, is called the *order-k Voronoi diagram* of S. If $k = 1$, this diagram is the well-known nearest-neighbor Voronoi diagram. For $k = n - 1$, it is the *farthest Voronoi diagram*, denoted by $\text{FVD}(S)$. Its *farthest regions* can also be defined directly as

$$\text{freg}(h) = \{p \in \mathbb{R}^d \mid s \in S \setminus \{h\} : \ d(p, h) > d(p, s)\}.$$

The i-skeleton of a Voronoi diagram is the union of all its j dimensional cells, where $j \leq i$.

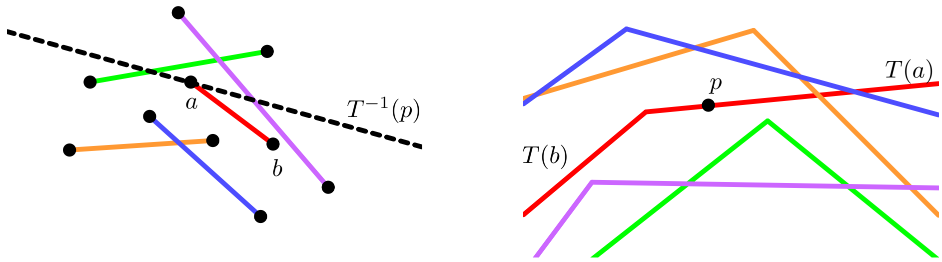

Figure 2 Point-hyperplane duality applied to segments: (left) Segments in primal space; and (right) their corresponding wedges in dual space

2.2 Point-Hyperplane Duality

Under the well-known standard point-hyperplane duality T in $\mathbb{R}^d$, a point $p \in \mathbb{R}^d$ is transformed to a nonvertical hyperplane $T(p)$, and vice versa. The transformation maps a point with coordinates $(p_1, p_2, ..., p_d)$ to the hyperplane $T(p)$ which satisfies the equation $x_d = -p_d + \sum_{i=1}^{d-1} p_i x_i$. The transformation is an involution, i.e., $T = T^{-1}$.

For a segment $s = uv$, the hyperplanes $T(u)$ and $T(v)$ partition the dual space into four *wedges*, among which the *lower wedge* (resp., the *upper wedge*) is the one that lies below (resp., above) both $T(u)$ and $T(v)$. The apex of the wedge is the intersection of $T(u)$ and $T(v)$.

Let S be a set of n segments, which corresponds in dual space to an arrangement of lower wedges. Let L_k be the kth level of that arrangement. Let p be a point on L_k, which touches the dual wedge of segment s, and let H be the set of segments whose wedge is below p. Then, the point p corresponds to a hyperplane $T^{-1}(p)$ which touches the segment s. The closed halfspace above $T^{-1}(p)$ has a non-empty intersection with the segments in H. The open halfspace above $T^{-1}(p)$ does not intersect any segment in $S \setminus H$, see Figure 2. We will use this property when we study the Gaussian map, which is defined in the next section.

2.3 Definition of the Gaussian Map

Let M be a cell complex in $\mathbb{R}^d$. We generalize the notion of the Gaussian map [3] which encodes information about the unbounded cells of M. This structure is of particular interest when all cells of M are unbounded. For example, all d-dimensional cells of the farthest Voronoi diagram of segments and/or lines are unbounded.

▶ **Definition 2.2.** A cell in M is called *unbounded in direction* $\vec{v}$ if it contains a ray with direction $\vec{v}$. The *Gaussian map* of M, denoted by GM(M), maps each cell in M to its unbounded directions, which are encoded on the unit sphere $\mathbb{S}^{d-1}$, see Figure 3. Let c be a cell of M; the set of directions, in which c is unbounded, is called the *region of* c on GM(M).

In this paper, we focus on cell complexes, such as the farthest Voronoi diagram and the order-k Voronoi diagram of lines and segments, where cells have unbounded direction and the Gaussian map implies a partition of $\mathbb{S}^{d-1}$. This induces a cell complex on $\mathbb{S}^{d-1}$. The collection of cells on the Gaussian map of a Voronoi diagram $\mathrm{VD}_k(S)$, which correspond to the same set of sites $H \subset S$, is called the *region of* H on GM($\mathrm{VD}_k(S)$). Note that a Gaussian map region of a cell can consist of several cells, *e.g.*, $\mathrm{reg}_2(\{s_3, s_4\})$ in Figure 3.

The order-k Voronoi diagram and its Gaussian map consist of vertices, edges, and cells in higher dimensions. The *complexity* of the order-k Voronoi diagram or the Gaussian map is the total number of all its cells of all dimensions.

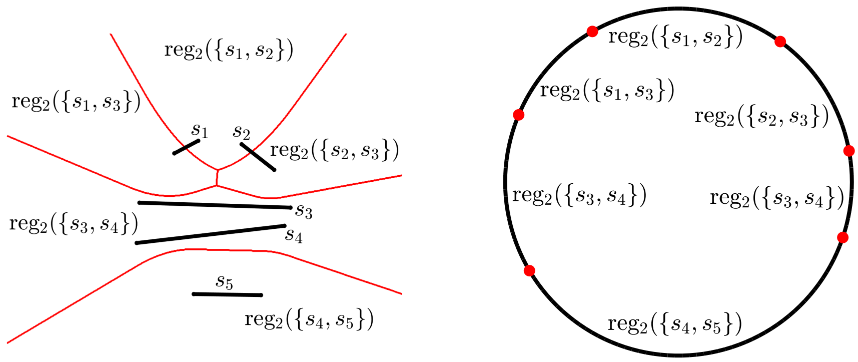

Figure 3 An order-2 Voronoi diagram $\mathrm{VD}_2(\{s_1, s_2, ..., s_5\})$ (left) and its Gaussian map (right).

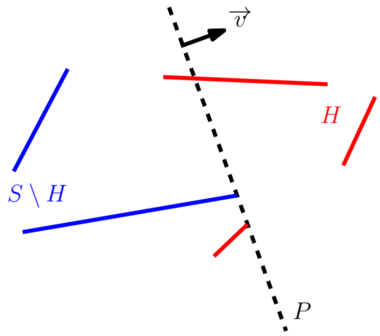

Figure 4 A supporting hyperplane P (in black, dashed) of sites H (in red) in direction $\vec{v}$.

3 Results

3.1 Supporting hyperplane

We first derive a characterization of the segments which induce an unbounded region of the order-k Voronoi diagram in a given direction.

▶ **Definition 3.1.** Let S be a set of segments, and let H be a subset of S. A hyperplane P is called a *supporting hyperplane* of H and S in direction $\vec{v}$ if

1. P is orthogonal to $\vec{v}$;
2. The closed halfspace P^+, bounded by P and unbounded in direction $\vec{v}$, has a non-empty intersection with each of the sites in H; and
3. The sites in $S \setminus H$ do not intersect the interior of P^+, and at least one site in $S \setminus H$ touches P.

Figure 4 illustrates a hyperplane supporting three segments.

▶ **Theorem 3.2.** *A set of segments H, with $|H| = k$, induces an unbounded region in direction $\vec{v}$ in the order-k Voronoi diagram of segments S, if and only if there exists a supporting hyperplane of H and S in direction $\vec{v}$.*

A supporting hyperplane, which touches i segments, corresponds to an unbounded cell of dimension $d - i + 1$ in the order-k Voronoi diagram. The proof of Theorem 3.2 and Theorem 3.7 is given in the full version.

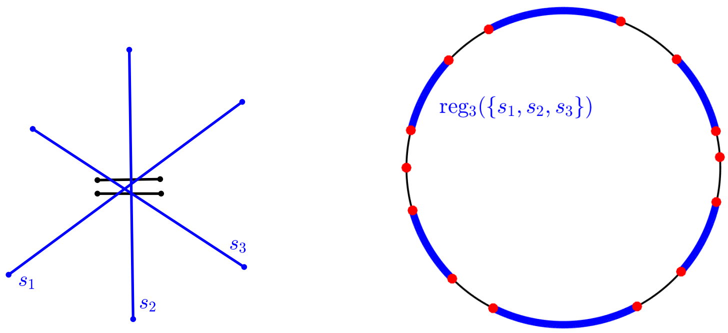

Figure 5 An instance of 5 segments (left), which has one region $\text{reg}_3(\{s_1, s_2, s_3\})$, shown in blue, on the Gaussian map of the order-3 Voronoi diagram (right) with high complexity.

3.2 Combinatorial Properties of the Gaussian Map

The next theorem provides a lower bound on the complexity of the Gaussian map of order-k Voronoi diagrams. This bound is meaningful if k is a function of n.

▶ **Theorem 3.3.** *Let S be a set of n line segments in $\mathbb{R}^d$. The complexity of a single region of the Gaussian map of the order-k Voronoi diagram is $\Omega(k^{d-1})$ in the worst-case.*

Proof. The bound is shown by a generalization of the examples provided for $\mathbb{R}^2$ [1, 11]. Place k long segments connecting almost antipodal points on a $(d-1)$-dimensional hypersphere and $n-k$ additional short segments near the center of the sphere, see Figure 5. Any $d-1$ tuple of long segments, together with a specific short segment, define a supporting hyperplane corresponding to an unbounded edge of the order-k Voronoi diagram. An unbounded edge of the diagram manifests as a vertex on $\text{GM}(\text{FVD}(S))$. All these vertices are on the boundary of the Gaussian map region of the long segments. ◀

▶ **Theorem 3.4.** *The complexity of the Gaussian map of the order-k Voronoi diagram of n segments in $\mathbb{R}^d$ is $O(\min(k, n-k)n^{d-1})$.*

Proof. In order to derive an upper bound on the complexity of the Gaussian map, we use the point-hyperplane duality transformation T, which establishes a 1-1 correspondence between the upper Gaussian map of the order-k Voronoi diagram and the kth level of the arrangement of d-dimensional wedges. The lower Gaussian map is constructed in the same manner. Each segment is mapped to a lower wedge in dual space, which is bounded by two half-hyperplanes. Let p be a point in dual space. Each wedge below p corresponds to a segment in primal space, which has a non-empty intersection with the open halfspace above hyperplane $T^{-1}(p)$. Each wedge touching p corresponds to a segment in primal space, which is touching the closed halfspace above hyperplane $T^{-1}(p)$. Each wedge above p corresponds to a segment in primal space, whose intersection with the closed halfspace above hyperplane $T(p)$ is empty. Therefore, every point on the kth level of the arrangement of the lower wedges corresponds to a hyperplane in primal space which supports k segments. The upper or lower envelope of those wedges, composed of two half-hyperplanes each, has complexity $O(n^{d-1})$ [6].

Using the bound on the lower envelope, we can now also bound the complexity of the $\leq k$-level of the arrangement of lower wedges. We apply a result by Clarkson and Shor [5] to derive a complexity of $O\left((k+1)^d \left(\frac{n}{k+1}\right)^{d-1}\right) = O\left(kn^{d-1}\right)$. We can derive a similar upper bound of $O((n-k)n^{d-1})$ by using the complexity of the upper envelope of lower wedges as a basis. The upper Gaussian map of the order-k Voronoi diagram corresponds to the $\leq k$-level of the lower wedges. Combining the two bounds completes the proof. ◀

Note that the bounds in Theorems 3.3 and 3.4 are tight for $n-k=O(1)$. In that case, the complexity of the Gaussian map of the order-k Voronoi diagram of n segments is $\Theta(n^{d-1})$ in the worst case.

3.3 Algorithm

▶ **Theorem 3.5.** *Let S be a set of n lines segments in $\mathbb{R}^3$. Then,* $\mathrm{GM}(\mathrm{FVD}(S))$ *can be constructed in worst-case optimal $O(n^2)$ time.*

Proof. We dualize each segment to derive a set of n lower wedges. The upper Gaussian map of the segments corresponds to the upper envelope of the lower wedges in dual space, as described in the proof of Theorem 3.4. The upper envelope of those wedges, each composed of two halfplanes, can be constructed in $O(n^2)$ time [6]. The lower Gaussian map can be constructed in the same manner. ◀

3.4 Properties of the order-k Voronoi Diagram

It was mentioned [3] that the complexity of the farthest Voronoi diagram in $\mathbb{R}^3$ is $O(n^{3+\varepsilon})$ (for any $\varepsilon > 0$), following the general bound on the upper envelope of "well-behaved" surfaces [12]. The same upper bound also holds for the order-k Voronoi diagram.

▶ **Corollary 3.6** (Theorem 3.3)**.** *The order-k Voronoi diagram of n segments in $\mathbb{R}^d$ has $\Omega(k^{d-1})$ complexity in the worst case. This bound becomes $\Omega(n^{d-1})$ for the farthest Voronoi diagram.*

Obviously, the stated lower bound for the order-k diagram is meaningful only for high values of k. Theorem 3.6 can be proven by showing that the number of vertices of the Gaussian map is $\Omega(k^{d-1})$ in the worst case. Each vertex of the Gaussian map corresponds to one edge in the Voronoi diagram. On the other hand, one edge of the diagram creates at most two vertices on the Gaussian map.

▶ **Theorem 3.7.** *The $(d-1)$-skeleton of the farthest Voronoi diagram of segments is connected.*

4 Lines and Combinations of Lines and Segments

For any set of n lines, there is a set of n segments, such that the Gaussian map of the order-k Voronoi diagram of the lines is the same as the one of the segments. Hence, Theorems 3.4, 3.5 extend to lines as sites. The same bounds hold for combined segments and lines as sites.

References

1 Franz Aurenhammer, Robert L. S. Drysdale, and Hannes Krasser. Farthest line segment Voronoi diagrams. *Information Processing Letters*, 100(6):220–225, 2006.

2 Franz Aurenhammer, Rolf Klein, and Der-Tsai Lee. *Voronoi diagrams and Delaunay triangulations*. World Scientific Publishing Company, 2013.

3 Gill Barequet and Evanthia Papadopoulou. On the farthest-neighbor Voronoi diagram of segments in three dimensions. In *10th International Symposium on Voronoi Diagrams in Science and Engineering (ISVD)*, pages 31–36. IEEE, 2013.

4 Bernard Chazelle. An optimal convex hull algorithm and new results on cuttings (extended abstract). In *32nd Annual Symposium on Foundations of Computer Science, San Juan, Puerto Rico*, pages 29–38. IEEE Computer Society, 1991.

5 Kenneth L. Clarkson and Peter W. Shor. Applications of random sampling in computational geometry, II. *Discrete & Computational Geometry*, 4(5):387–421, 1989.

6 Herbert Edelsbrunner, Leonidas J. Guibas, and Micha Sharir. The upper envelope of piecewise linear functions: Algorithms and applications. *Discrete & Computational Geometry*, 4:311–336, 1989.

7 Herbert Edelsbrunner and Raimund Seidel. Voronoi diagrams and arrangements. *Discrete & Computational Geometry*, 1:25–44, 1986.

8 Hazel Everett, Daniel Lazard, Sylvain Lazard, and Mohab Safey El Din. The Voronoi diagram of three lines. *Discrete & Computational Geometry*, 42(1):94–130, 2009.

9 Victor Klee. On the complexity of d-dimensional Voronoi diagrams. *Archiv der Mathematik*, 34(1):75–80, 1980.

10 Joseph S. B. Mitchell and Joseph O'Rourke. Computational geometry column 42. *International Journal of Computational Geometry & Applications*, 11(5):573–582, 2001.

11 Evanthia Papadopoulou and Maksym Zavershynskyi. The higher-order Voronoi diagram of line segments. *Algorithmica*, 74(1):415–439, 2016.

12 Micha Sharir. Almost tight upper bounds for lower envelopes in higher dimensions. *Discrete & Computational Geometry*, 12:327–345, 1994.