On the 2-Colored Crossing Number*

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Abstract

Let D be a straight-line drawing of a graph where every edge is colored with one of two possible colors. The rectilinear 2-colored crossing number of D is the minimum number of crossings between edges of the same color, taken over all possible colorings of D. We show lower and upper bounds on the rectilinear 2-colored crossing number for the complete graph K_n. Moreover, for fixed drawings of K_n we give bounds on the relation between its rectilinear 2-colored crossing number and its rectilinear crossing number.

1 Introduction

In any drawing D of a non-planar graph G in the plane, two of its edges will cross. From both a theoretical and practical point of view it is of interest to study the minimum number of pairs of edges that cross in any drawing of G. This is known as the *crossing number* $\text{cr}(G)$ of G. There are many variants on crossing numbers. In this paper we focus on a variant mixing two of them: the *biplanar crossing number* and the *rectilinear crossing number*.

The *biplanar crossing number* of a graph G, $\text{cr}_2(G)$, is the minimum of $\text{cr}(G_1) + \text{cr}(G_2)$ over all graphs G_1 and G_2 whose union is G. This parameter was introduced by Owens [12]. For a survey on biplanar crossing numbers of graphs see [9, 10].

A *straight-line drawing* of G is a drawing D of G in the plane in which the vertices are drawn as points in general position and the edges are drawn as straight line segments. We identify the vertices and edges of the underlying abstract graph with the corresponding ones in the straight-line drawing. The *rectilinear crossing number* of G, $\overline{\text{cr}}(G)$, is the minimum number of pairs of edges that cross in any straight-line drawing of G. Of special relevance is $\overline{\text{cr}}(K_n)$, the rectilinear crossing number of the complete graph on n vertices. The current best published bounds on $\overline{\text{cr}}(K_n)$ are $0.379972\binom{n}{4} < \overline{\text{cr}}(K_n) < 0.380473\binom{n}{4} + \Theta(n^3)$ [3, 11]. The upper bound has been improved in an upcoming paper [4] to $\overline{\text{cr}}(K_n) < 0.3804493\binom{n}{4} + \Theta(n^3)$.

A 2-*edge-coloring* of a drawing D of a graph is an assignment of one of two possible colors to every edge of D. The *rectilinear 2-colored crossing number* of a graph G, $\overline{\text{cr}}_2(G)$, is the minimum number of monochromatic crossings (pairs of edges of the same color that cross)

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in a 2-edge-colored straight-line drawing of G. This parameter was introduced before and called the *geometric 2-planar crossing number* [13]. We prefer our terminology because the term k-planar is extensively used in graph drawing with a different meaning.

In this paper we focus on the case where G is the complete graph K_n, and we prove the following lower and upper bounds on $\overline{\text{cr}}_2(K_n)$:

$$1/33\binom{n}{4} + \Theta(n^3) < \overline{\text{cr}}_2(K_n) < 0.11798016\binom{n}{4} + \Theta(n^3).$$

Combining the lower bound on $\overline{\text{cr}}(K_n)$ and the upper bound on $\overline{\text{cr}}_2(K_n)$ implies that asymptotically, $\overline{\text{cr}}_2(K_n)/\overline{\text{cr}}(K_n) \leq 0.31049652 + o(1)$.

Note that drawings with few crossings don't necessarily admit a coloring with few monochromatic crossings. This observation motivates the following question: Given a fixed straight-line drawing D of K_n, what is the ratio between the number of monochromatic crossings for the best 2-edge-coloring of D and the number of (uncolored) crossings in D? A simple probabilistic argument shows that this ratio is at most $1/2$. We show that for sufficiently large n, it is less than $1/2 - c$ for some positive constant c.

In a slight abuse of notation, we denote with $\overline{\text{cr}}(D)$ the number of pairs of edges in D that cross and call it the rectilinear crossing number of D. The (rectilinear) 2-colored crossing number of a straight-line drawing D, $\overline{\text{cr}}_2(D)$, is then the minimum of $\overline{\text{cr}}(D_1) + \overline{\text{cr}}(D_2)$, over all straight-line drawings D_1 and D_2 whose union is D. For a given 2-edge-coloring χ of D, we denote with $\overline{\text{cr}}_2(D, \chi)$ the number of monochromatic crossings in D. Thus, $\overline{\text{cr}}_2(D)$ is the minimum of $\overline{\text{cr}}_2(D, \chi)$ over all 2-edge-colorings χ of D.

2 Lower bounds on $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D)$

In this section we study the extreme values that $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D)$ can attain for straight-line drawings D of K_n. In the full version of this paper we explore certain classes of straight-line drawings of K_n. Among others, we show that there exist classes of drawings D of K_n for which $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) \leq 1/3 + o(1)$ and that for convex straight-line drawings of K_n it holds that $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) = 3/8 - o(1)$.

Using a simple probabilistic argument it can be shown that $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) \leq 1/2$ for every straight-line drawing D of K_n. This lower bound can be improved for any such drawing.

▶ **Theorem 2.1.** *There exists a positive integer n_0 and a positive constant c such that for any straight-line drawing D of the complete graph K_n on $n \geq n_0$ vertices, $\overline{\text{cr}}_2(D)/\overline{\text{cr}}(D) \leq 1/2 - c$.*

The proof, based on the positive Fraction Erdős-Szekeres theorem [7], can be found in the full version of this paper.

3 Upper bounds on $\overline{\text{cr}}_2(K_n)$

For the rectilinear crossing number $\overline{\text{cr}}(K_n)$ the best upper bound [4] comes from finding examples of straight-line drawings of K_n with few crossings (for a small value of n) which are then used as a seed for the duplication process in [2, 3]. In this section, we prove that a more involved but similar approach can be adopted for the two-colored case.

Throughout this section P is a set of m points in general position in the plane. Let p be a point in P. Given a 2-edge-coloring χ of the edges of the straight-line drawing of K_m that P induces, we denote by $L(p)$ and $S(p)$ the edges incident to p of the larger and smaller color class at p, respectively. An edge e incident to p is called a *χ-halving edge of p* if the number of edges of $L(p)$ to the right of the line ℓ_e spanned by e (and directed from q to p) and the number of edges of $L(p)$ to the left of ℓ_e differ by at most one. An assignment between the points of P and its χ-halving edges is called a *χ-halving matching of P*.

▶ **Theorem 3.1.** *Let P be a set of m points in general position with a two-coloring χ of the edges of the straight-line drawing of K_m it induces. If P has a χ-halving matching, then*

$$\overline{\text{cr}}_2(K_n) \leq \frac{24A}{m^4}\binom{n}{4} + \Theta(n^3)$$

where A is a rational number that depends on P, χ, and the χ-halving matching of P.

Proof. First we describe a process to obtain from P a set Q of $2m$ points, a 2-edge-coloring χ' of the of the straight-line drawing of K_{2m} that Q induces, and a χ'-halving matching for Q. The set Q is constructed as follows. Let p be a point in P and $e = (p, q)$ its χ-halving edge in the matching. We add to Q two points p_1, p_2 placed along the line spanned by e and in a small neighborhood of p such that: (i) if f is an edge different from e that is incident to p, then p_1 and p_2 lie on different sides of the line spanned by f; (ii) if f is an edge different from e that is not incident to p, then p_1 and p_2 lie on the same side of the line spanned by f as p; and (iii) the point p_1 is farther away from q than p_2. The set Q has $2m$ points and the above conditions ensure that they are in general position.

Next, we define a coloring χ' and a χ'-halving matching for Q. For every edge (p, q) of P, color the four edges $(p_i, q_j), i, j \in \{1, 2\}$ with the same color as (p, q). Hence the only edges remaining to be colored are the edges (p_1, p_2) between the duplicates of a point $p \in P$. Let ℓ_e be the line spanned by e and directed from q to p and let q_1, q_2 be the points that originated from q, such that q_1 lies to the left of ℓ_e and q_2 lies to the right of ℓ_e. We assume that the colors are red and blue and that the larger color class at p is blue.

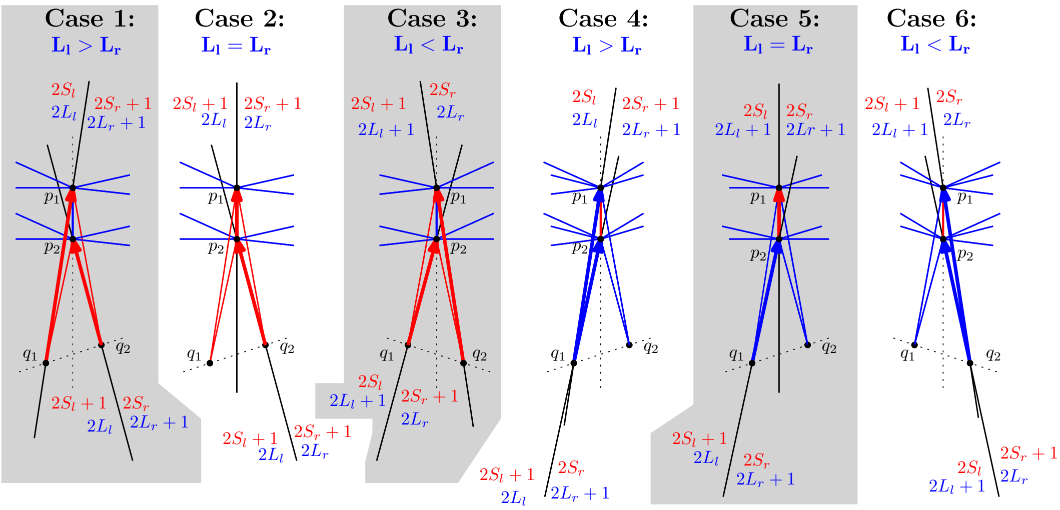

Figure 1 The cases in the duplication process of Theorem 3.1 when the larger color class at p is blue. The edge e of P has color red in the first 3 cases and color blue in the last 3 cases. The dotted lines represent the lines spanned by the χ-halving edges for P. The thick edges represent the χ'-halving edges for Q, where the arrow points to the point it is matched to. L_l and L_r (S_l and S_r) represent the number of blue (red) edges at p to the left and right of ℓ_e, respectively. The number of colored edges on each side of the lines spanned by thick edges indicate the resulting numbers of red and blue edges to the left and right of the χ'-halving edges for Q.

There are six cases in which p can fall. They are all shown in Figure 1, which depicts among others the color that the edge (p_1, p_2) receives and that the edges matched to p_1 and p_2 are indeed χ'-halving edges in each case.

Note that no point in Q falls in Case 5. From now on, we assume that no point in P falls in Case 5. Our goal is to iterate the duplication process and obtain a bound on $\overline{\text{cr}}_2(K_n)$. Let $k \geq 1$ be an integer and let (Q_k, χ_k) be the pair obtained by iterating the duplication process k times. We claim the following on $\overline{\text{cr}}_2(Q_k, \chi_k)$, the number of monochromatic crossings in the straight-line 2-edge colored drawing of K_n induced by Q_k and χ_k:

▶ **Claim.** *After k iterations of the duplication process, the following holds*

$$\overline{\text{cr}}_2(Q_k, \chi_k) = A \cdot 2^{4k} + B \cdot 2^{3k} + C \cdot 2^{2k} + D \cdot 2^k$$

where A, B, C and D are rational numbers that depend on P and its χ-halving matching.

The proof of this claim can be found in the full version of this paper. Letting $n = 2^k m$:

$$\overline{\text{cr}}_2(K_n) \leq \overline{\text{cr}}_2(Q_k, \chi_k) = \frac{24A}{m^4}\binom{n}{4} + \Theta(n^3)$$

which proves the theorem when n is of the form $2^k m$. The proof for $2^k m < n < 2^{k+1} m$ follows from showing that $\overline{\text{cr}}(K_n)$ is an increasing function. ◀

3.1 Small configurations

The previous section implies that for a large number of vertices we can obtain straight-line drawings of the complete graph with a reasonable small 2-colored crossing number from *good* sets of constant size. Thus, in this section we describe how to obtain those small good sets.

Similar as in [4] we combine different methods to obtain straight-line drawings of the complete graph with low 2-colored crossing number. The overall approach is to apply three different methods in alternating order: we start with a known set, apply the duplication process from Theorem 3.2 to obtain a larger set, locally optimize it to get a better set, find good subsets, locally optimize them, duplicate the resulting sets and so on.

The currently best (w.r.t. to the crossing constant, see below) straight-line drawing D with 2-edges coloring χ we found[1] has $n = 135$ vertices, a 2-colored crossing number of $\overline{\text{cr}}_2(D, \chi) = 1470756$, and contains a χ-halving matching.

3.2 Rectilinear 2-colored crossing constant

Let $\overline{\text{cr}}_2$ be the *rectilinear 2-colored crossing constant*, that is, the constant such that the best straight-line drawing of K_n for large values of n has at most $\overline{\text{cr}}_2\binom{n}{4}$ monochromatic crossings. Its existence follows from showing that $\lim_{n\to\infty} \overline{\text{cr}}_2(K_n)/\binom{n}{4}$ exits and is a positive number.

From the previous results in this section we can derive an upper bound for the 2-colored crossing constant from a given set of constant size with a small 2-colored crossing number:

Plugging the values of the set of 135 points obtained in the last section into Theorem 3.2 (after duplicating once to get rid of Case 5) we get the upper bound of $\overline{\text{cr}}_2 < 0.11798016$.

▶ **Theorem 3.2.** *The 2-colored crossing constant satisfies* $\overline{\text{cr}}_2 \leq \frac{182873519}{1550036250} < 0.11798016$.

[1] The interested reader can get a file with the coordinates of the points, the colors of the edges, and a χ-halving matching from `http://www.crossingnumbers.org/projects/monochromatic/sets/n135.php`.

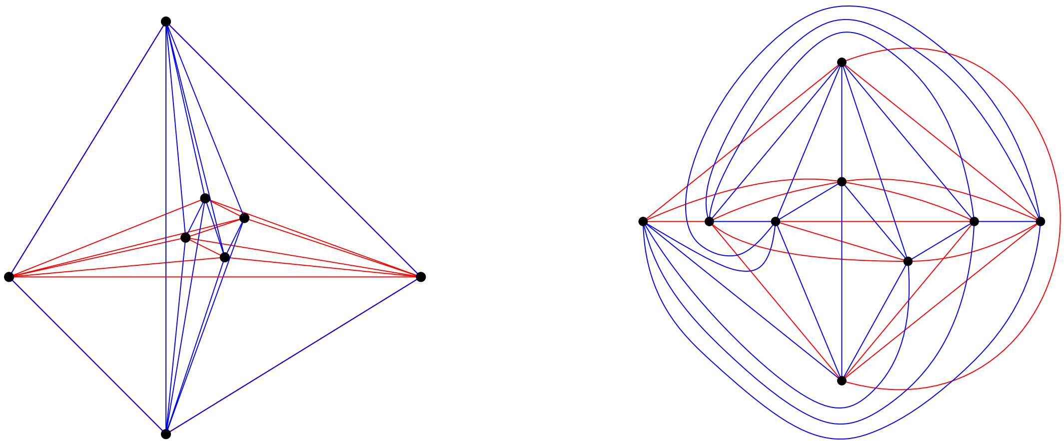

Figure 2 Left: a 2-colored rectilinear drawing of K_8 without monochromatic crossings. Right: a 2-colored drawing of K_9 with only one monochromatic (red) crossing.

In [3] a lower bound of $\overline{\text{cr}} \geq \frac{277}{729} > 0.37997267$ has been shown for the rectilinear crossing constant. We can thus give an upper bound on the asymptotic ratio between the best 2-colored straight-line drawing of K_n and the best straight-line drawing of K_n of $\overline{\text{cr}}_2/\overline{\text{cr}} \leq 0.31049652$.

4 Lower bounds on $\overline{\text{cr}}_2(K_n)$ and $\text{cr}_2(K_n)$

The following result shows that from the 2-colored rectilinear crossing number of small sets we can obtain lower bounds for larger sets.

▶ **Lemma 4.1.** *Let $\overline{\text{cr}}_2(\hat{n}) = \hat{c}$ for some $\hat{n} \geq 4$. Then for $n > \hat{n}$ we have $\overline{\text{cr}}_2(K_n) \geq \frac{24\hat{c}}{\hat{n}(\hat{n}-1)(\hat{n}-2)(\hat{n}-3)}\binom{n}{4}$ which implies $\overline{\text{cr}}_2 \geq \frac{24\hat{c}}{\hat{n}(\hat{n}-1)(\hat{n}-2)(\hat{n}-3)}$*

Proof. Every subset of $\hat{n}$ points of K_n induces a drawing with at least $\hat{c}$ crossings, and thus we have $\hat{c}\binom{n}{\hat{n}}$ crossings in total. In this way every crossing is counted $\binom{n-4}{\hat{n}-4}$ times. This results in a total of $\frac{24\hat{c}}{\hat{n}(\hat{n}-1)(\hat{n}-2)(\hat{n}-3)}\binom{n}{4}$ crossings. ◀

With a strategy based on the intersection graph of a given straight-line drawing, we have been able to determine all the 2-colored crossing numbers of all straight-line drawings of K_9 and prove that $\overline{\text{cr}}_2(K_9) = 2$. More details about this strategy can be found in the full version. Using Lemma 4.1 for $\hat{n} = 9$ and $\hat{c} = 2$ we get a bound of $\overline{\text{cr}}_2 \geq 1/63$. Repeating the process of computing lower bounds for sets of small cardinality we checked all order types of size 11 [5]. We obtained $\overline{\text{cr}}_2(K_{11}) = 10$ and by Lemma 4.1 this gives the even better bound of $\overline{\text{cr}}_2 \geq 1/33$.

4.1 Staight-line versus general drawings

The best straight-line drawings of K_n with $n \leq 8$ have no monochromatic crossing, see Figure 2 left for a straight-line 2-colored crossing-free drawing of K_8. In [13], Section 3, the authors claim that up to now no graph was known were the k-colored crossing number was strictly smaller than the rectilinear k-colored crossing number for any $k \geq 2$. From the previous section we know that $\overline{\text{cr}}_2(K_9) = 2$. Inspecting rotation systems for $n = 9$ [1] which have the minimum number of 36 crossings, we have been able to construct a drawing of K_9 which has only one monochromatic crossing, see Figure 2 right. As the graph thickness of

K_9 is 3 [8, 14], we can not draw K_9 with just two colors without monochromatic crossings. Thus, the biplanar crossing number for K_9 is 1 and thus strictly smaller than $\overline{\text{cr}}_2(K_9) = 2$.

5 Conclusion and open problems

In this paper we have shown lower and upper bounds on the rectilinear 2-colored crossing number for K_n as well as its relation to the rectilinear crossing number for fixed drawings of K_n. Besides improving the given bounds, some open problems arise from our work. The first question is how fast we can compute the best edge-coloring of a given rectilinear drawing of K_n. A second question is on the structure of 2-colored crossing minimal sets. For the rectilinear crossing number it is known that optimal sets have a triangular convex hull [6]. For $n = 8, 9$ we have optimal sets with 3 and 4 extreme points, but so far all minimal sets for $n \geq 10$ have a triangular convex hull.

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