# Rigid Graphs that are Movable*

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**Abstract**

A graph is called movable if there exists a proper flexible labeling, i.e., an edge labeling such that there are infinitely many injective realizations of the graph in the plane, counted modulo rigid motions, such that the distances between adjacent vertices equal the labels. Of special interest is the class of generically rigid graphs that are movable due to a non-generic proper flexible labeling. We introduce two methods for investigating possible proper flexible labelings. The first one is based on restrictions to 4-cycles and gives an easy classification of all but one non-bipartite 6 and 7-vertex graphs in the class. Using our second method, we prove that every proper flexible labeling of this one graph forces the vertices in its only 3-cycle to be collinear.

## 1 Introduction

In Rigidity Theory, realizations of a graph in $\mathbb{R}^2$ are required to be such that the distances of adjacent vertices are equal to a given labeling of edges by positive real numbers. Such a labeling is called *(proper) flexible* if the number of (injective) realizations, counted modulo rigid transformations, is infinite. Otherwise, the labeling is called rigid. We call a graph movable if there is a proper flexible labeling.

A result of Pollaczek-Geiringer [6], rediscovered by Laman [5], shows that a graph is *generically rigid*, i.e., a generic realization defines a rigid labeling, if and only if the graph contains a Laman subgraph with the same set of vertices. A graph $G = (V_G, E_G)$ is called Laman if $|E_G| = 2|V_G| - 3$, and $|E_H| \leq 2|V_H| - 3$ for all subgraphs $H$ of $G$. Hence, every graph that is not spanned by a Laman graph is movable.

A natural question is which generically rigid graphs are movable, due to a non-generic proper flexible labeling. For instance, two ways of making the bipartite Laman graph $K_{3,3}$ movable were given by Dixon more than one hundred years ago [2, 9, 7], and it was proven much later in 2007 that these give all proper flexible labelings [8]. In [3], we provide a combinatorial characterization of existence of a flexible labeling, not necessarily proper: it exists if and only if the graph has a so called *NAC-coloring* (see Figure 1). In [4] we classified all movable graphs up to 8 vertices.

The movable graphs up to 7 vertices with special properties are listed in Figure 2. They do not have a degree two vertex, they are spanned by a Laman graph and they are maximal with respect of being subgraph of a movable graph with the same number of vertices. These are the interesting ones since a graph with a degree two vertex $v$ is movable if and only if the graph with $v$ removed is movable; and a spanning subgraph of a movable graph is movable.

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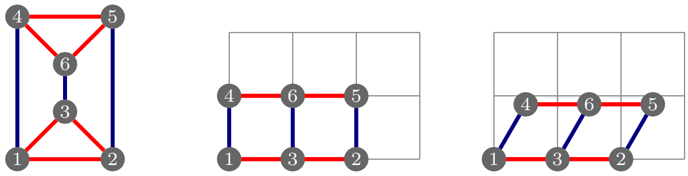



**Figure 1** Motion from a NAC-coloring.

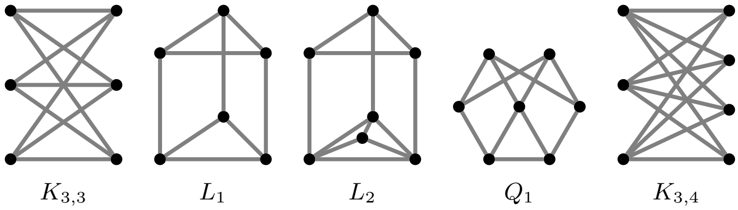



**Figure 2** Movable graphs with a spanning Laman subgraph and at most 7 vertices.

The classification of proper flexible labelings of $K_{3,3}$ and $K_{3,4}$ is known. In this paper, we introduce methods allowing a step towards classification for the other graphs. The first one is based on restriction to 4-cycles and gives an easy proof that $L_1$ and $L_2$ have only one type of proper flexible labelings (Section 3). The proper flexible labeling provided in [4] makes the vertices of the triangle in $Q_1$ collinear. Our second method allows to show that this is always the case (Section 4). The method uses leading coefficients of certain Laurent series and is based on tropicalization ideas presented in [1].

## 2 Preliminaries

We recall the concept of a NAC-coloring, which is a basic object for our further considerations.

▶ **Definition 2.1.** Let $G$ be a graph. A coloring of edges $\delta\colon E_G \to \{\text{blue, red}\}$ is called a *NAC-coloring*, if it is surjective and for every cycle $C$ in $G$, either all edges of $C$ have the same color, or $C$ contains at least 2 edges in each color. If NAC-colorings $\delta, \overline{\delta}$ of $G$ are such that $\delta(e) = \text{blue} \iff \overline{\delta}(e) = \text{red}$ for all $e \in E_G$, then they are called *conjugated*.

We remark that NAC stands for “No Almost Cycle”, i.e., there is no cycle with all but one edges having the same color. Next, we summarize the concepts from Rigidity Theory.

▶ **Definition 2.2.** Let $G$ be a graph such that $|E_G| \geq 1$ and let $\lambda\colon E_G \to \mathbb{R}_+$ be an edge labeling of $G$. A map $\rho = (\rho_x, \rho_y)\colon V_G \to \mathbb{R}^2$ is a *realization of $G$ compatible with $\lambda$* if $\|\rho(u) - \rho(v)\| = \lambda(uv)$ for all edges $uv \in E_G$. The labeling $\lambda$ is called *(proper) flexible* if the number of (injective) realizations of $G$ compatible with $\lambda$ up to direct Euclidean isometries is infinite. A graph is called *movable* if it has a proper flexible labeling.

We fix an edge $\bar{u}\bar{v}$ by setting $x_{\bar{u}} = y_{\bar{u}} = y_{\bar{v}} = 0$ and $x_{\bar{v}} = \lambda_{\bar{u}\bar{v}}$ for removing rotations and translations (reflection on x-axis is kept). Then the realizations are the solutions of the system of equations for lengths $\lambda_{uv} = \lambda(uv)$ and coordinates $(x_u, y_u)$ for $u \in V_G$ given by

$$(x_u - x_v)^2 + (y_u - y_v)^2 = \lambda_{uv}^2 \quad \text{for all } uv \in E_G \setminus \{\bar{u}\bar{v}\}. \tag{1}$$

For working with function fields, we consider the irreducible components of the solution set.

▶ **Definition 2.3.** Let $\lambda$ be a flexible labeling of $G$. We say that $\mathcal{C}$ is an *algebraic motion of* $(G, \lambda)$, if it is an irreducible algebraic curve of realizations compatible with $\lambda$, such that $\rho(\bar{u}) = (0, 0)$ and $\rho(\bar{v}) = (\lambda_{\bar{u}\bar{v}}, 0)$ for all $\rho \in \mathcal{C}$.

In order to link an algebraic motion with a NAC-coloring, we define the following functions:

▶ **Definition 2.4.** Let $\lambda$ be a flexible labeling of a graph $G$. Let $F(\mathcal{C})$ be the complex function field of an algebraic motion $\mathcal{C}$ of $(G, \lambda)$. For every $u, v \in V_G$ such that $uv \in E_G$, we define $W_{u,v}, Z_{u,v} \in F(\mathcal{C})$ by $W_{u,v} = (x_v - x_u) + i(y_v - y_u)$ and $Z_{u,v} = (x_v - x_u) - i(y_v - y_u)$.

Using (1), we have $W_{\bar{u},\bar{v}} = \lambda_{\bar{u}\bar{v}}, Z_{\bar{u},\bar{v}} = \lambda_{\bar{u}\bar{v}}$ and $W_{u,v}Z_{u,v} = \lambda_{uv}^2$ for all $uv \in E_G$. Moreover, the following equations hold for every cycle $(u_0, u_1, \ldots, u_n, u_{n+1} = u_0)$ in $G$:

$$\sum_{i=0}^{n} W_{u_i,u_{i+1}} = 0 \quad \text{and} \quad \sum_{i=0}^{n} Z_{u_i,u_{i+1}} = 0\,. \tag{2}$$

▶ **Definition 2.5.** Let $\mathcal{C}$ be an algebraic motion of $(G, \lambda)$. A NAC-coloring $\delta$ of $G$ is called *active* if there exists a valuation $\nu$ of $F(\mathcal{C})$ and $\alpha \in \mathbb{Q}$ such that $\delta(uv) = \text{red}$ if and only if $\nu(W_{u,v}) > \alpha$ for all $uv \in E_G$. The set of all active NAC-colorings is denoted by $\mathrm{NAC}_G(\mathcal{C})$.

We remark that $\mathrm{NAC}_G(\mathcal{C})$ is closed under conjugation and independent of the fixed edge [4].

## 3 Restriction to 4-cycles

One way of investigating proper flexible labelings is to look at restrictions to 4-cycle subgraphs, since the active NAC-colorings of an algebraic motion of $(C_4, \lambda)$ can be described:

▶ **Lemma 3.1.** *Let $\mathcal{C}$ be an algebraic motion of a 4-cycle graph $C_4$ with a labeling $\lambda$. Table 1 summarizes* $\mathrm{NAC}_{C_4}(\mathcal{C})$ *depending on $\lambda$ using the notation depicted on Figure 3.*

**Proof.** Explicit computation — solving the system of equations for $W_{u,v}$ and $Z_{u,v}$ and determining valuations giving active NAC-colorings. ◀

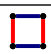
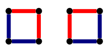
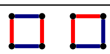
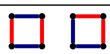
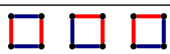
| Quadrilateral | Motion | active NAC-colorings | | Equations |
|---|---|---|---|---|
| Rhombus | parallel |  | $O$ | $\left\{\begin{array}{c}\lambda_{12} = \lambda_{23} = \\ = \lambda_{34} = \lambda_{14}\end{array}\right\}$ |
| | degenerate #1 resp. #2 |  resp.  | $L$ resp. $R$ | |
| Parallelogram | parallel |  | $O$ | $\left\{\begin{array}{c}\lambda_{12} = \lambda_{34}, \\ \lambda_{23} = \lambda_{14}\end{array}\right\}$ |
| | antiparallel |  | $L, R$ | |
| Deltoid 1 | nondegenerate |  | $O, R$ | $\left\{\begin{array}{c}\lambda_{12} = \lambda_{14}, \\ \lambda_{23} = \lambda_{34}\end{array}\right\}$ |
| | degenerate |  | $L$ | |
| Deltoid 2 | nondegenerate |  | $O, L$ | $\left\{\begin{array}{c}\lambda_{12} = \lambda_{23}, \\ \lambda_{34} = \lambda_{14}\end{array}\right\}$ |
| | degenerate |  | $R$ | |
| General | |  | $O, L, R$ | otherwise |

**Table 1** Active NAC-colorings of possible motions of a $(C_4, \lambda)$.

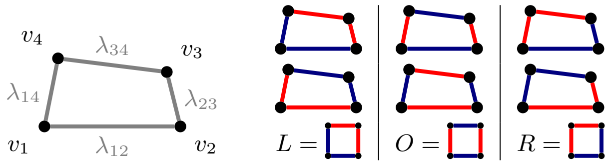
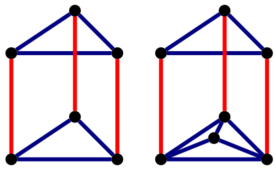



**Figure 3** Labeling of the 4-cycle $C_4$ and notation for conjugated NAC-colorings.

Assuming an algebraic motion $\mathcal{C}$ of a graph $G$, the active NAC-colorings of the projection of $\mathcal{C}$ to a 4-cycle subgraph of $G$ are precisely the restrictions of the active NAC-colorings of $\mathcal{C}$. It is well known that $L_1$ and $L_2$ are movable by making the vertical edges in Figure 2 parallel and same lengths. Looking at 4-cycles gives easily that this is the only option:

▶ **Corollary 3.2.** *If $\lambda$ is a proper flexible labeling of $L_1$, resp. $L_2$, then every 4-cycle that is colored non-trivially by $\delta_1$, resp. $\delta_2$, (see Figure 4) is a parallelogram.*

**Proof.** Since $\delta_i$ is the only NAC-coloring of $L_i$ modulo conjugation, it is the only active NAC-coloring in every algebraic motion. The restriction of $\delta_i$ to each non-trivially colored 4-cycle is of type O (). According to Table 1 it must be in a parallel motion. ◀



$\delta_1$ $\delta_2$

**Figure 4** The only NAC-colorings of $L_1$ and $L_2$ modulo conjugation.

## 4 Leading coefficients system

If a movable graph $G$ is spanned by a Laman graph, the edge lengths $\lambda_{uv}$ must be non-generic. We introduce a method deriving some algebraic equation(s) for $\lambda_{uv}$, which must be satisfied if a NAC-coloring $\delta$ is active due to a valuation $\nu$ under a certain assumption.

Let an edge $\bar{u}\bar{v}$ be fixed. Then, $\nu(W_{\bar{u},\bar{v}}) = 0$. We can assume that $\delta(\bar{u}\bar{v}) = \text{blue}$, otherwise we replace $\delta$ by its conjugated NAC-coloring. There must be a red edge $\hat{u}\hat{v}$ with $\nu(W_{\hat{u},\hat{v}}) = \alpha > 0$. We assume that $\nu$ yields no other active NAC-coloring besides $\delta$. This assumption implies that $\{\nu(W_{u,v}) \colon uv \in E_G\} = \{0, \alpha\}$. Notice that if $\nu$ yields another active NAC-coloring $\delta'$, then the set $\{(\delta(e), \delta'(e)) \colon e \in E_{Q_1}\}$ has 3 elements.

There are Laurent series parametrizations of $W_{u,v}$ and $Z_{u,v}$ such that $\operatorname{ord}(W_{u,v}) = \nu(W_{u,v})$ and $\operatorname{ord}(Z_{u,v}) = \nu(Z_{u,v})$. Since $\nu(W_{\hat{u},\hat{v}}) = \alpha > 0$, we can reparametrize so that $W_{\hat{u},\hat{v}} = \lambda_{\hat{u}\hat{v}} t$. Hence, there is $\varepsilon > 0$ such that the parametrizations are

$$\begin{aligned}
&W_{\bar{u},\bar{v}} = \lambda_{\bar{u}\bar{v}}\,, \quad Z_{\bar{u},\bar{v}} = \lambda_{\bar{u}\bar{v}}\,, \quad W_{\hat{u},\hat{v}} = \lambda_{\hat{u}\hat{v}} t\,, \quad Z_{\hat{u},\hat{v}} = \lambda_{\hat{u}\hat{v}} t^{-1}\,,\\
&W_{u,v} = w_{uv} + O(t^{\varepsilon})\,, \qquad Z_{u,v} = z_{uv} + O(t^{\varepsilon}) \qquad \text{for } uv \in E_G \setminus \{\bar{u}\bar{v}\},\ \delta(uv) = \text{blue}\,,\\
&W_{u,v} = w_{uv} t + O(t^{1+\varepsilon})\,, \quad Z_{u,v} = z_{uv} t^{-1} + O(t^{-1+\varepsilon}) \qquad \text{for } uv \in E_G \setminus \{\hat{u}\hat{v}\},\ \delta(uv) = \text{red}\,.
\end{aligned}$$

The constraints from edge lengths give $w_{uv} z_{uv} = \lambda_{uv}^2$ for all $uv \in E_G \setminus \{\bar{u}\bar{v}, \hat{u}\hat{v}\}$.

For every cycle $C = (u_0, u_1, \dots, u_n, u_{n+1} = u_0)$ in $G$, the first equation in (2) gives

$$\sum_{\substack{i\in\{0,\dots,n\}\\ \delta(u_iu_{i+1})=\text{red}}} (w_{u_iu_{i+1}}t + O(t^{1+\varepsilon})) + \sum_{\substack{i\in\{0,\dots,n\}\\ \delta(u_iu_{i+1})=\text{blue}}} (w_{u_iu_{i+1}} + O(t^{\varepsilon})) = 0\,.$$

Comparing leading coefficients gives $\sum w_{u_iu_{i+1}} = 0$, where the sum is over all $i \in \{0,\dots,n\}$ such that $\delta(u_iu_{i+1}) = \text{blue}$ if there exists a blue edge, or over all edges in $C$ otherwise. Similarly, we obtain an equation in $z_{uv}$'s from the second equation in (2).

From the obtained equations from all cycles and edge lengths, we eliminate $w_{uv}$ and $z_{uv}$ for all $uv \in E_G \setminus \{\bar{u}\bar{v}, \hat{u}\hat{v}\}$ using Gröbner basis. If the graph $G$ is spanned by a Laman graph, we expect to get some algebraic equation(s) in $\lambda_{uv}$ for $uv \in E_G$. We use the described procedure for the graph $Q_1$. The NAC-colorings of $Q_1$, modulo conjugation, are in Figure 5.

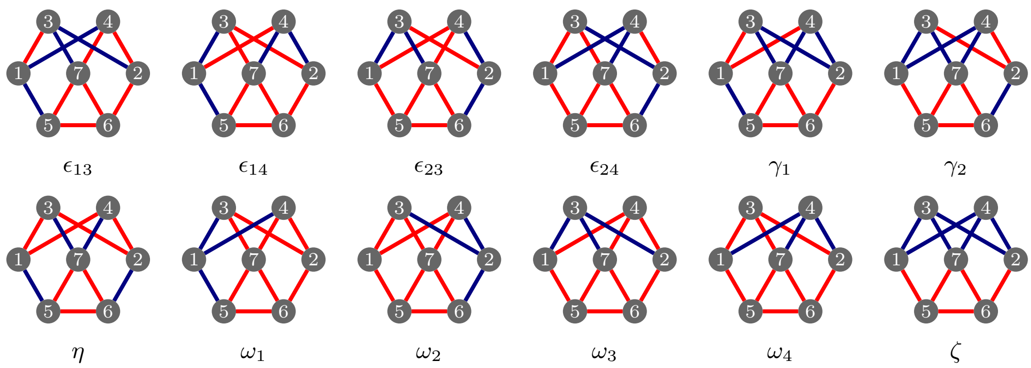



**Figure 5** NAC-colorings of the graph $Q_1$.

▶ **Lemma 4.1.** *Let $\mathcal{C}$ be an algebraic motion of $(Q_1, \lambda)$. If $\eta \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, resp. $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, then the vertices of the triangle (5,6,7) are collinear, or $\lambda_{24} = \lambda_{23}$ and $\lambda_{14} = \lambda_{13}$, resp. $\lambda_{26} = \lambda_{67}$ and $\lambda_{24} = \lambda_{47}$.*

**Proof.** The procedure described at the beginning of this section can be used for $\eta$, since for every other NAC-coloring $\delta$ of $Q_1$, the set $\{(\eta(e), \delta(e)) : e \in E_{Q_1}\}$ has 4 elements. The equation $\lambda_{57}^2 r^2 + \lambda_{67}^2 s^2 + \left(\lambda_{56}^2 - \lambda_{57}^2 - \lambda_{67}^2\right) rs = 0$ is obtained, where $r = \lambda_{24}^2 - \lambda_{23}^2$ and $s = \lambda_{14}^2 - \lambda_{13}^2$. Considering the equation as a polynomial in $r$, the discriminant is $(\lambda_{56} + \lambda_{57} + \lambda_{67})(\lambda_{56} + \lambda_{57} - \lambda_{67})(\lambda_{56} - \lambda_{57} + \lambda_{67})(\lambda_{56} - \lambda_{57} - \lambda_{67})s^2$. But this is always non-positive from the triangle inequality. Hence, the triangle must be degenerate, or $s = 0$. If $s = 0$, then also $r = 0$ and the statement follows. The proof for $\epsilon_{13}$ is similar. ◀

We prove the following lemma by combining the previous one with the restrictions to 4-cycles.

▶ **Lemma 4.2.** *Let $\mathcal{C}$ be an algebraic motion of $(Q_1, \lambda)$ such that $\lambda$ is a proper flexible labeling and the vertices 5,6,7 are not collinear. If $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, then $\epsilon_{14}, \gamma_1, \gamma_2, \omega_1, \omega_2, \omega_4, \zeta \notin \mathrm{NAC}_{Q_1}(\mathcal{C})$, either $\epsilon_{23} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$ or $\epsilon_{24} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, and $\eta \in \mathrm{NAC}_{Q_1}(\mathcal{C})$.*

**Proof.** By the assumption and Lemma 4.1, if $\epsilon_{13} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$, then $\lambda_{26} = \lambda_{67}$ and $\lambda_{24} = \lambda_{47}$. But then the 4-cycle (2,4,7,6) is a deltoid or rhombus. Hence, the restriction of any active NAC-coloring to (2,4,7,6) cannot be of type R by Lemma 3.1, i.e., $\epsilon_{14}, \gamma_1, \omega_4, \zeta \notin \mathrm{NAC}_{Q_1}(\mathcal{C})$ by Table 2. Since the 4-cycle (2,3,7,4) cannot be an antiparallelogram, there must be an active NAC-coloring whose restriction is of type O, namely, $\epsilon_{23} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$ or $\epsilon_{24} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$. Since $\epsilon_{13}$ excludes $\epsilon_{14}$ to be active, $\epsilon_{23}$ excludes $\epsilon_{24}$ by graph symmetry. Therefore, either

$\epsilon_{23} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$ or $\epsilon_{24} \in \mathrm{NAC}_{Q_1}(\mathcal{C})$. By the symmetric approach to the fact that $\epsilon_{13}$ excludes $\gamma_1$, we also get that both $\epsilon_{23}$ and $\epsilon_{24}$ prohibit $\gamma_2$ to be active. Since the 4-cycle (2,4,7,6), resp. (2,3,7,6), is not an antiparallelogram, there must be an active NAC-coloring restricting to O, namely $\epsilon_{24}$ or $\eta$, resp. $\epsilon_{23}$ or $\eta$ ($\gamma_2$ is already excluded). In the combination with the previous, we can conclude that $\eta \in \mathrm{NAC}_{Q_1}(\mathcal{C})$. Therefore, $\lambda_{24} = \lambda_{23}$ and $\lambda_{14} = \lambda_{13}$ by Lemma 4.1. This shows that the 4-cycle (1,3,2,4) is a deltoid or rhombus which prohibits L. Thus, $\omega_1, \omega_2 \notin \mathrm{NAC}_{Q_1}(\mathcal{C})$. ◀

| 4-cycle | $\epsilon_{13}$ | $\epsilon_{14}$ | $\epsilon_{23}$ | $\epsilon_{24}$ | $\gamma_1$ | $\gamma_2$ | $\eta$ | $\omega_1$ | $\omega_2$ | $\omega_3$ | $\omega_4$ | $\zeta$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $(1,3,2,4)$ | O | O | O | O | L | L | S | L | L | R | R | S |
| $(1,3,7,4)$ | O | O | R | R | L | S | L | L | S | R | R | S |
| $(2,3,7,4)$ | R | R | O | O | S | L | L | S | L | R | R | S |
| $(1,3,7,5)$ | O | L | R | S | O | R | O | L | S | R | S | R |
| $(1,4,7,5)$ | L | O | S | R | O | R | O | L | S | S | R | R |
| $(2,3,7,6)$ | R | S | O | L | R | O | O | S | L | R | S | R |
| $(2,4,7,6)$ | S | R | L | O | R | O | O | S | L | S | R | R |

**Table 2** Types of NAC-colorings of $Q_1$ restricted to 4-cycles using the notation from Figure 3 and S meaning all edges have the same color.

We conclude that the triangle (5,6,7) in $Q_1$ is always degenerate.

▶ **Theorem 4.3.** *If $\mathcal{C}$ is an algebraic motion of $Q_1$ with infinitely many injective realization, then the vertices 5,6 and 7 are always collinear.*

**Proof.** If no $\epsilon_{ij}$ is active, then the 4-cycles (1,3,2,4), (1,3,7,4) and (2,3,7,4) are all antiparallelograms. But this is not possible for injective realizations. Hence, by symmetry we can assume w.l.o.g. that $\epsilon_{13}$ is active. Suppose by contradiction that the triangle (5,6,7) is not degenerated. By Lemma 4.2 (and its symmetric version for $\epsilon_{24}$), the only possibilities for $\mathrm{NAC}_{Q_1}(\mathcal{C})$ are $\{\epsilon_{13}, \epsilon_{23}, \eta\}$, $\{\epsilon_{13}, \epsilon_{23}, \eta, \omega_3\}$ and $\{\epsilon_{13}, \epsilon_{24}, \eta\}$. By careful determination of the types of all 4-cycles for each of these sets of active NAC-colorings, the equality of lengths always contradicts injective realizations. For instance for $\mathrm{NAC}_{Q_1}(\mathcal{C}) = \{\epsilon_{13}, \epsilon_{23}, \eta\}$, all edges among 1,2,3,4 and 7 would have to have the same lengths, which is not possible. ◀

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