Erdős-Szekeres-Type Games*

Oswin Aichholzer[1], José-Miguel Díaz-Báñez[2], Thomas Hackl[1], David Orden[3], Alexander Pilz[1], Inmaculada Ventura[2], and Birgit Vogtenhuber[1]

1 **Institute of Software Technology, Graz University of Technology, Austria.**
`[oaich|thackl|apilz|bvogt]@ist.tugraz.at`

2 **Department of Applied Mathematics II, University of Seville, Spain.**
`[dbanez|iventura]@us.es`

3 **Departamento de Física y Matemáticas, Universidad de Alcalá, Spain.**
`david.orden@uah.es`

Abstract

We consider several combinatorial games, inspired by the Erdős-Szekeres theorem that states the existence of a convex k-gon in every sufficiently large point set. Two players take turns to place points in the Euclidean plane and the game is over as soon as the first k-gon appears. In the Maker-Maker setting the player who placed the last point wins, while in the Avoider-Avoider version this player loses. Combined versions like Maker-Breaker are also possible. Moreover, variants can be obtained by considering that (1) the points to be placed are either uncolored or bichromatic, (2) both players have their own color or can play with both colors, (3) the k-gon must be empty of other points, or (4) the k-gon has to be convex.

1 Introduction

A central topic in combinatorial game theory are sequential games with perfect information. These are often two-player games that have positions, in which the players take turns changing these positions (in a defined way) to eventually achieve a specific winning position. Perfect information means that the state of the game (the current position and usually also the history of all moves so far) and the set of all possible moves is known to both players at any time. This class includes Chess or Go, but also easy-to-analyze games like Tic-tac-toe. The formal analysis of concrete games sometimes reveals challenging mathematical problems while still having substantial recreational value. Along these lines, we study a class of combinatorial games related to a well-known result in combinatorics, the Erdős-Szekeres theorem.

▶ **Theorem 1.1** (Erdős and Szekeres [5]). *For every integer $k \geq 3$, there exists an $n(k)$ s.t. any set of at least $n(k)$ points in general position has a k-element subset in convex position.*

Decades later, Erdős [4] posed the problem of determining the smallest integer $h(k)$, if it exists, such that any set S of at least $h(k)$ points in general position contains a convex k-hole, that is, a convex k-gon that does not contain any point of S in its interior. Finding

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exact lower bounds on the number of points to guarantee the existence of convex k-gons (see [6, 12, 16]) and k-holes (see [10, 11, 9, 15, 17]) has been a long line of research.

A simple polygon spanned by points of different color classes is called *monochromatic* if all its vertices have the same color. Two results of interest for our games are the following.

▶ **Theorem 1.2** (Devillers et al. [3]). *Every bichromatic set of* 9 *points in general position contains an empty monochromatic triangle.*

▶ **Theorem 1.3** (Devillers et al. [3]). *There exist arbitrarily large bichromatic point sets in general position without any convex monochromatic 5-hole.*

The question of whether every bichromatic point set of sufficiently many points contains a convex monochromatic 4-hole is still open, see [2, 8, 18]. However, we have:

▶ **Theorem 1.4** (Aichholzer et al. [1]). *Every bichromatic set of at least* 5044 *points contains a (possibly non-convex) monochromatic* 4*-hole.*

In this context, the following combinatorial game comes to mind immediately: Two players alternately place one point at a time in the plane until a k-gon appears, with the points required to be pairwise distinct and in general position. Different games arise depending on whether the points are colored or not. We will call the players *player A* and *player B* if both players can play both colors (say, red and green), and call them *Red* and *Green* if each player is only allowed to use a specific color. Players A and Red will always start a game, and are thus called *first player*. Consequently, we call players B and Green *second player*. In this paper we study the following scenarios:

1. The Erdős-Szekeres Avoider-Avoider game (ESAA), in which the goal is to avoid the formation of a convex monochromatic k-gon/k-hole. The player who places the last point that results in a convex k-gon/k-hole loses.
2. The Erdős-Szekeres Maker-Maker game (ESMM), in which the goal is to build a convex monochromatic k-gon/k-hole and the player who places the last point forming the convex k-gon/k-hole wins.
3. The Erdős-Szekeres Maker-Breaker game (ESMB), in which the Red player aims to make a red convex/general k-hole and the goal of the Green player is to prevent this.

Preliminaries. In combinatorial game theory, a sequential game is a game where one player performs their action before the next player performs theirs. The players act in turns, where one action of a player is called a *ply*. The term "ply" is used to avoid confusion when one might otherwise use the term move (or turn). All considered or constructed point sets will be in general position, i.e., no three points of a set are on a common line.

The Avoider-Avoider *uncolored* game for *convex* 5*-gons* and 5*-holes* was shown by Kolipaka and Govindarajan [13] to be a win of B. We provide a shorter proof via a simple strategy for B in Section 2. In Section 3, the *bichromatic* ESAA and ESMM games are addressed both for A–B and Red–Green players, as well as the ESMB game for Red–Green players.

2 Uncolored variants

In the setting in which the points do not have different colors, the symmetric game types Avoider-Avoider and Maker-Maker immediately come to mind. Observe that the Maker-Maker variant for convex k-gons/k-holes corresponds to the Avoider-Avoider variant for $k-1$. Here, we thus discuss the Avoider-Avoider variant. (For holes, it would also make sense to consider Maker–Breaker, at least for $k \geq 7$, as there are point sets without convex 7-hole [11].)

▶ **Game 1.** *Two players A and B alternate in placing a single (uncolored) point in the plane with the restriction that no three points are on a line. The player who places a point resulting in a convex k-gon/k-hole loses the game.*

The cases for $k \in \{3, 4\}$ are trivial. For $k = 5$, Kolipaka and Govindarajan [13] showed that player B wins at the 9th ply. We provide a simpler strategy leading to a shorter proof (the original proof results in a paper of 28 pages), which works for both 5-holes and 5-gons.

▶ **Theorem 2.1** (Kolipaka and Govindarajan [13]). *For $k = 5$, player B wins Game 1 at ply 9.*

Proof (sketch). We start by showing the result for 5-holes. Player B will use the *point reflection strategy*: After player A placed the first point, player B chooses a different point acting as center of symmetry, around which B mimics the plies of player A. We can argue that after 6 plies, this strategy leads to mostly equivalent configurations consisting of a parallelogram with two points inside (having central symmetry); see Figure 1 (a) and (b).

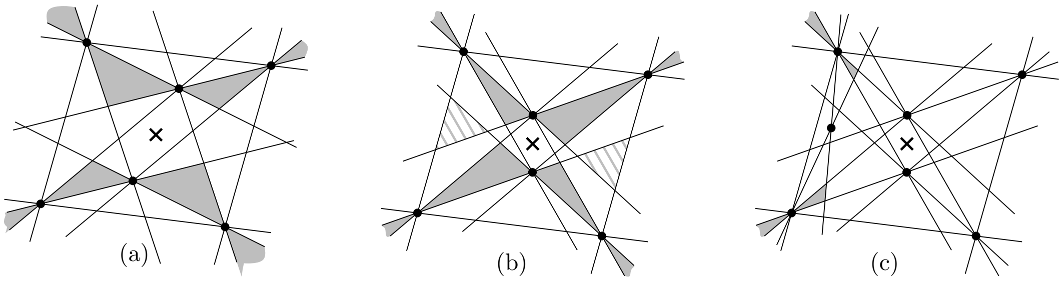

Figure 1 Configurations of six points with central symmetry and no convex 5-hole/5-gon.

The next point of player A has to be placed in a gray or hatched region, as otherwise a convex 5-hole occurs. If A plays in any gray region, then after ply 8 we get the situation depicted in Figure 2 (a). If A plays in a hatched region of Figure 1 (b), then the setting of Figure 2 (b) shows up. In both cases there is no convex 5-hole. Moreover, no additional point can be placed without generating a convex 5-hole (although there exist sets of 9 points in general position without any convex 5-hole [10]). Thus player B wins in ply 9.

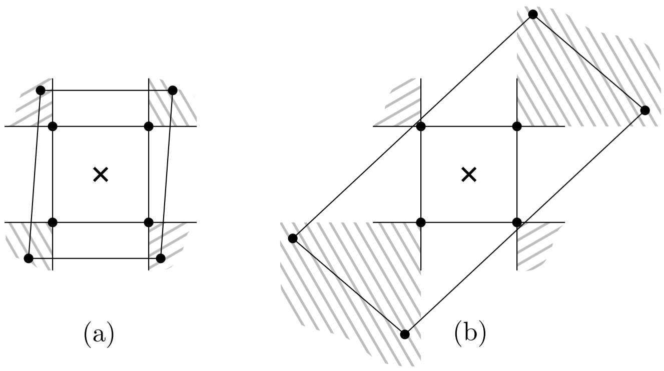

Figure 2 Any configuration without convex 5-hole must have the four points of the outer parallelogram in the indicated regions. Situation (b) cannot occur when avoiding a convex 5-gon.

For 5-gons, the arguments are analogous up to ply 6 and also until the end if player A places a point in a gray region of Figure 1 (a) or (b) in ply 7. So assume that player A places a point in a hatched region of Figure 1(b), as depicted in Figure 1(c). Then in ply 8

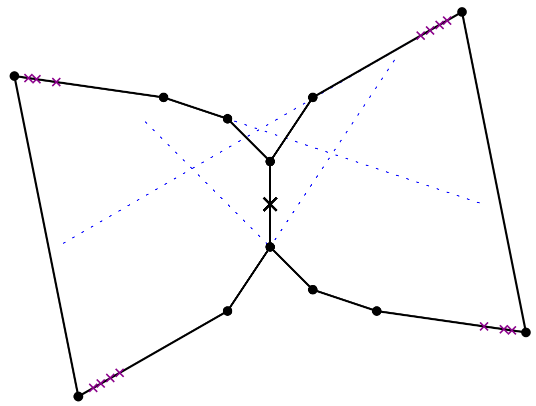

Figure 3 Counterexample for the point reflection strategy for k-holes for $k = 7$. The two points closest to the center are the last to be placed, where the last one causes the formation of two 7-holes. The small purple crosses indicate how to generalize this example for $k \geq 7$.

player B would produce a 5-gon with the point reflection strategy and hence has to make a different move; see again Figure 2 (b). However, in this case Player B can put a point in any of the grey regions indicated in Figure 1 (c). As every set of nine points in general position contains a 5-gon [12], player B wins in ply 9. ◀

For any even k, the point reflection strategy cannot work for player B: if player A places all points in convex position, then B creates a convex k-hole. For any odd k, a convex k-gon can never be centrally symmetric. Hence, in centrally symmetric point sets they come in pairs. Figure 3 shows that for odd $k > 5$ the strategy does not work either.

3 Bichromatic variants

In bichromatic games, either each player is assigned a color which they can use (then we call the players Red and Green), or both players (called A and B) may use both colors. In either case, the goal is to make or avoid a monochromatic k-gon or k-hole. A hole must not contain any points in its interior, regardless of their color. We consider the game with players Red and Green only for k-holes, as for k-gons the different colors do not influence each other.

3.1 Avoider–Avoider

We first consider the two-colored Avoider–Avoider setting in both versions, players A–B and Red–Green. We show that, for any version, the second player can avoid losing. For triangles, we provide upper bounds on the number of plies until the second player wins.

▶ **Game 2.** *In both versions (players Red–Green and A–B), the players alternate in placing a single point of one of two colors in the plane, avoiding collinear point triples. The goal for each player is to avoid the formation of a (general or convex) monochromatic k-hole. The player who placed the last point forming such a k-hole loses the game.*

The next theorem follows from the point reflection strategy with color-inversion.

▶ **Theorem 3.1.** *Players Green and B can avoid losing the respective variants of Game 2.*

By Theorem 1.3, there exist arbitrarily large sets without convex monochromatic 5-holes. It is open whether every sufficiently large bichromatic set has a convex monochromatic 4-hole [1], so we do not know if the game is finite for convex monochromatic k-holes, $k \geq 4$.

For $k = 3$, every bichromatic point set of at least 9 points contains an empty monochromatic triangle (Theorem 1.2). Thus, Theorem 3.1 implies a 9-ply winning strategy for the second player. We further show the following (proofs are omitted due to space constrains).

▶ **Theorem 3.2.** *Player B can win the bichromatic A–B Avoider–Avoider empty monochromatic triangle game at latest after the 9th ply, even if the point set must be in strong general position*[1]*. Player A can prevent to lose before the 9th ply.*

▶ **Theorem 3.3.** *Player Green can win the bichromatic Red–Green Avoider–Avoider empty monochromatic triangle game at latest after the 7th ply.*

For $k = 4$, the largest known point set not containing any convex monochromatic 4-hole has 46 points [14]. It can be modified (changing also the order type) to be centrally symmetric, with symmetry pairs having inverted colors [7]. From the latter set it follows that, if the second player mirrors the moves of the first player, the game might take at least 47 plies. For general monochromatic 4-holes, there is a set of 22 points not containing any of them [7], but without such a special symmetry. Both sets have the same number of red and green points. As the second player always has a strategy to not lose, the following questions arise.

▶ **Question 1.** *Does the second player have a winning strategy for monochromatic 4-holes by placing points centrally symmetric and color-inverted? What about for k-holes for $k > 4$?*

▶ **Question 2.** *Does every large enough centrally symmetric color-inverted bichromatic point set contain a convex monochromatic 4-hole?*

3.2 Maker–Maker

In the Maker–Maker variant the goal is to be the first to obtain a monochromatic k-hole. For players A and B, both must avoid a monochromatic $(k-1)$-hole; for convex k-holes, this is also sufficient. For general 4-holes, player A always makes a monochromatic triangle (empty or not) in the 5th ply. Then, player B can add another point inside this triangle as in Figure 4 (left) and produce a non-convex monochromatic 4-hole, i.e., wins in ply 6. For $k = 5$ we do not know any upper bound on the number of plies, convex or general.

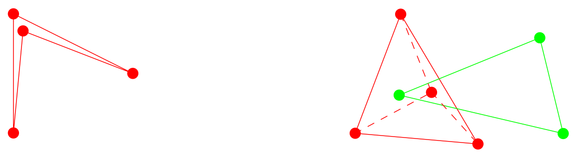

Figure 4 Non-convex red 4-hole and Red–Green Maker–Maker after 7 plies for convex red 4-hole.

For players Red–Green, we argue that Red wins for $k = 3, 4$. A win in ply 5 for $k = 3$ is obvious. For $k = 4$, Red makes an empty red triangle Δ_R with the first three red points. In ply 6 player Green must place a green point inside Δ_R. Red can make two interior-disjoint empty red triangles by placing the fourth red point inside Δ_R and also inside the green triangle, see Figure 4 (right). This is already a general red 4-hole. For convex 4-holes, Green

[1] No two supporting lines of distinct point pairs are parallel; no three intersect in a common point; etc.

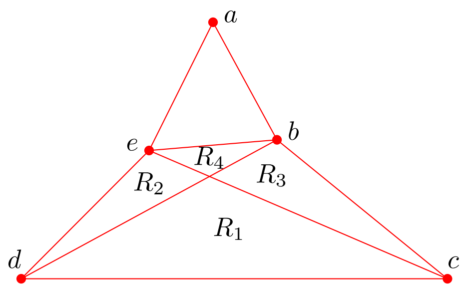

Figure 5 A can build a red general 5-hole in 9 plies.

can neither extend the green triangle nor block both of the two empty red triangles. Thus, Red wins in ply 9. We also show the following.

▶ **Lemma 3.4.** *Player Red can always build a red (general) 5-hole as depicted in Figure 5 in 9 plies, that is, with the minimum number of 5 red points.*

▶ **Question 3.** *For players Red–Green, is there a winning strategy for convex monochromatic 5-holes? For which of the two players? How about $k > 5$?*

3.3 Red–Green Maker–Breaker

In this section we consider an asymmetric game, where the two players have different goals.

▶ **Game 3.** *Two players Red and Green alternate in placing a single point of their color, avoiding collinearities. The goal for player Red is to make a red k-hole. Player Green wants to block Red from doing so. Green k-holes do not matter.*

The following result can be shown using the idea sketched in Figure 6.

▶ **Theorem 3.5.** *In a Red–Green Maker–Breaker game, the Maker can always build a red (general) k-hole by placing k points.*

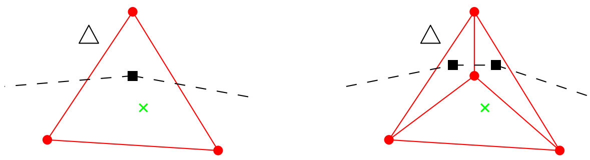

Figure 6 The Maker builds a chain of empty triangles (indicated by dashed lines and squares). When the Breaker places a green point (cross) inside such a triangle △, the Maker places a red point inside △, thereby repairing the chain and increasing its length by one.

From Lemma 3.4 we derive the following statements.

▶ **Proposition 3.6.** *The Maker can always build a red convex 4-hole by placing 5 red points.*

▶ **Theorem 3.7.** *The Maker can always build a red convex 5-hole by placing 8 red points.*

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