Linear-size farthest color Voronoi diagrams: conditions and algorithms*

Ioannis Mantas[1], Evanthia Papadopoulou[1], Vera Sacristán[2], and Rodrigo I. Silveira[2]

1 **Faculty of Informatics, Università della Svizzera italiana, Lugano, Switzerland**
`{ioannis.mantas,evanthia.papadopoulou}@usi.ch`

2 **Departament de Matemàtiques, Universitat Politècnica de Catalunya, Barcelona, Spain**
`{vera.sacristan,rodrigo.silveira}@upc.edu`

Abstract

The *farthest-color Voronoi diagram* (FCVD) is a *farthest-site* Voronoi diagram defined on a family of m clusters (sets) of points in the plane. Its combinatorial complexity in the worst case is $\Theta(mn)$, where n is the total number of points. In this paper we give structural properties of the FCVD and list sufficient conditions under which this diagram has $O(n)$ combinatorial complexity. For such cases we present efficient construction algorithms.

1 Introduction

The *Voronoi diagram* is a well-known geometric partitioning structure, defined by a set of simple geometric objects in a space, called sites. The ordinary *(nearest-neighbor) Voronoi diagram* of a set of points in two dimensions is a subdivision of the plane into maximal regions such that all points in one region share the same nearest site. In the *farthest-site Voronoi diagram*, points in a single region have the same farthest site. Many generalizations of this simple concept have been considered for different types of sites, metrics and spaces. For a comprehensive list of results see [2].

We are interested in *color Voronoi diagrams*, where each site is a *cluster* (a set) of points in $\mathbb{R}^2$, identified by a distinct color. The distance between a point $x \in \mathbb{R}^2$ and a cluster P is realized by the nearest point in P, i.e., $d_c(x, P) = \min_{p \in P} d(x, p)$. The *nearest-color Voronoi diagram (NCVD)* of a family $\mathcal{P}$ of clusters, is a *min-min* diagram that can be easily derived from the ordinary Voronoi diagram of all points in $\mathcal{P}$: the region of a cluster P is the union of the Voronoi regions of points belonging to P (see Fig. 1a). Its farthest counterpart, the *farthest-color Voronoi diagram (FCVD)* of $\mathcal{P}$ is a *max-min* diagram and its properties have not been extensively looked into (see Fig. 1b).

The FCVD was first studied by Huttenlocher et al. [9], showing that the combinatorial complexity of the diagram in the worst case is $\Omega(mn)$ and $O(mn\alpha(mn))$, where m is the number of clusters and n is the overall number of points. This was later settled to $\Theta(mn)$ by Abellanas et al. [1]. Using a geometric transformation in 3D, the diagram can be computed in $O(mn \log n)$ time [9]: for every cluster P, each point in the plane is lifted in 3 dimensions, with height equal to the distance from the nearest point in P, yielding a surface; the upper

* This research and in particular the work of I. M. and E. P. has been supported in part by the Swiss National Science Foundation, under the DACH project VORONOI++, SNF 200021E-154387. R. S. was supported by MINECO through the Ramón y Cajal program. V. S. and R. S. were also supported by projects MINECO MTM2015-63791-R and Gen. Cat. 2017SGR1640. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

35th European Workshop on Computational Geometry, Utrecht, The Netherlands, March 19–20, 2019.
This is an extended abstract of a presentation given at EuroCG'19. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

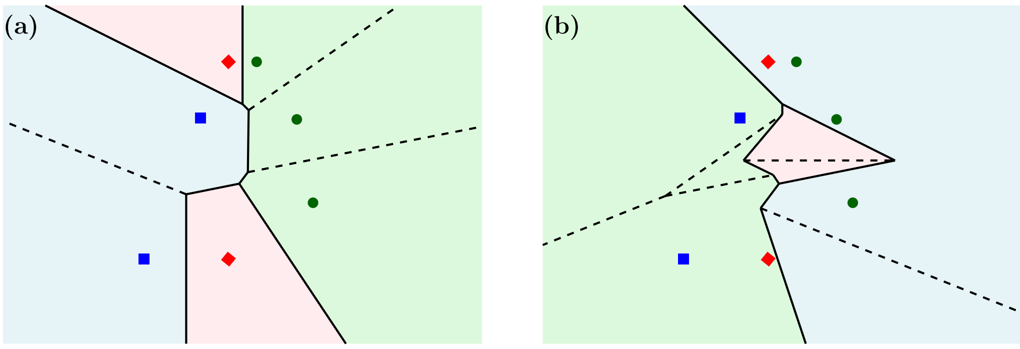

Figure 1 A family $\mathcal{P}$ of clusters along with (a) NCVD($\mathcal{P}$) and (b) FCVD($\mathcal{P}$).

envelope of these surfaces projected back onto the plane gives the FCVD. Instances of linear-size diagrams have been considered by Bae [3], Claverol et al. [6] and Iacono et al. [10]. Applications of the FCVD include facility location problems [1], variants of the *Steiner tree* problem [4], sensor deployment problems [13] and finding *stabbing circles* for line segments [6].

Closely related to the FCVD is the *Hausdorff Voronoi diagram* (HVD) of a family of point clusters. The HVD is a *min-max* diagram: the distance from a point $x \in \mathbb{R}^2$ to a cluster is the farthest distance, $d_f = \max_{p \in P} d(x,p)$, and the plane is subdivided into maximal regions with the same nearest cluster. The HVD has been extensively studied, see e.g. [8, 15], and many algorithmic paradigms have been considered for its construction, see e.g. [7, 11, 15, 16]. Interestingly, the algorithm presented in [8] can be adapted to also yield an $O(n^2)$-time algorithm for the FCVD. This has been remarked in [6] for point clusters of cardinality two. In the worst case, this is optimal as the diagram may have complexity $\Theta(n^2)$. However, the algorithm remains $\Theta(n^2)$ even if the diagram has only $O(n)$ structural complexity.

In this work, we study structural properties of the FCVD, give sufficient conditions under which the diagram has $O(n)$ structural complexity and present efficient algorithms to construct it when these conditions are met.

2 Definitions and basic properties

Let $\mathcal{P} := \{P_1, ..., P_m\}$ be a family of m *clusters of points* in $\mathbb{R}^2$, where no two clusters share a point. We assume that $m > 1$ and let $\sum_{i=1...m} |P_i| = n$. We define the following diagrams.

▶ **Definition 1.** The *nearest color Voronoi diagram (NCVD)* of $\mathcal{P}$ is the subdivision of $\mathbb{R}^2$ into *nearest color Voronoi regions*. The *nearest color Voronoi region* of a cluster $P_i \in \mathcal{P}$ is $n_c reg(P_i) = \{x \in \mathbb{R}^2 \mid d_c(x, P_i) < d_c(x, P_j)\ \forall P_j \in \mathcal{P},\ j \neq i\}$.

▶ **Definition 2.** The *farthest color Voronoi diagram (FCVD)* of $\mathcal{P}$ is the subdivision of $\mathbb{R}^2$ into *farthest color Voronoi regions*. The *farthest color Voronoi region* of a cluster $P_i \in \mathcal{P}$ is $f_c reg(P_i) = \{x \in \mathbb{R}^2 \mid d_c(x, P_i) > d_c(x, P_j)\ \forall P_j \in \mathcal{P},\ j \neq i\}$.

A region $f_c reg(P_i)$ may consist of several maximally connected components, called *faces*. Faces of $f_c reg(P_i)$ are further subdivided by the ordinary Voronoi diagram of P_i, which is denoted $\mathrm{Vor}(P_i)$. This is called the *internal subdivision* of a face. For $p \in P_i : f_c reg(p) = \{x \in f_c reg(P_i) \mid d(x,p) < d(x,q)\ \forall q \in P_i \setminus \{p\}\}$. A region $f_c reg(p)$ may have several faces.

▶ **Definition 3.** Given two clusters P_i, P_j, their *color bisector* is the locus of points equidistant from the two clusters, that is, $b_c(P_i, P_j) = \{x \in \mathbb{R}^2 \mid d_c(x, P_i) = d_c(x, P_j)\}$.

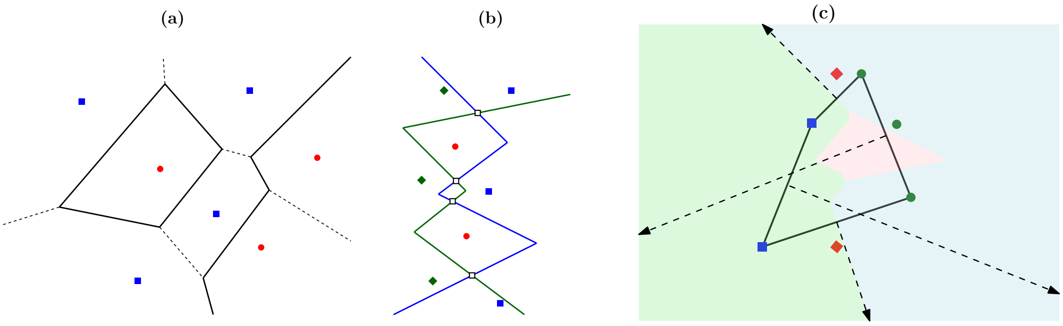

Figure 2 (a) A bisector consisting of a cycle and a chain. (b) Two bisectors sharing a site intersecting linearly many times. (c) Hull of the clusters in Fig. 1 and the associated normal vectors.

Bisector $b_c(P_i, P_j)$ is a subgraph of the Voronoi diagram $\mathrm{Vor}(P_i \cup P_j)$. It is a collection of edge-disjoint cycles and unbounded chains of total complexity $O(|P_i| + |P_j|)$, which is tight in the worst case (see Fig. 2a).

We refer to edges of the FCVD belonging to color bisectors as *pure edges*, and to edges or vertices of the internal subdivisions as *internal*. Voronoi vertices incident to three color bisectors are called *pure vertices*, and vertices incident to two color bisectors and one internal edge are called *mixed vertices*. See Fig. 3 for an illustration of these features.

The following lemma characterizes the structure of farthest color regions.

▶ **Lemma 2.1.** *A face f of $f_c reg(P_i)$ satisfies:*

1. *If f is bounded, its internal subdivision is a tree whose leaves are mixed vertices on ∂f.*
2. *If f is unbounded, its internal subdivision is a (possibly empty) forest, where each tree has exactly one unbounded edge and its remaining leaves are mixed vertices on ∂f.*

The boundary of a face $f_c reg(p), p \in P_i$, is a sequence of convex chains (as seen from p).

We use a refinement of the FCVD derived by the *visibility decomposition*, defined analogously to [16]: For each region $f_c reg(p)$ and for each pure or mixed vertex u on $\partial f_c reg(p)$, draw the portion of the line through p and u that lies inside $f_c reg(p)$ (see Fig. 3).

The *cluster hull*, for short *hull*, of a family of point clusters is a closed (not necessarily simple) polygonal chain that characterizes the unbounded faces of the FCVD and the HVD. We review the definition from [16], see Fig. 2c.

▶ **Definition 4.** Given a family of clusters $\mathcal{P}$, a point $p \in P_i$ is a *hull vertex* if p admits a supporting line l, such that P_i lies completely on one of the two halfplanes defined by l and the other one intersects every cluster $P_j \in \mathcal{P} \setminus \{P_i\}$. A *hull edge* is a segment connecting two hull vertices $p \in P_i, q \in P_j$ such that the line through p, q leaves P_i and P_j on one halfplane, while the other halfplane intersects all other clusters in $\mathcal{P}$. Such an edge is associated with a normal vector in the direction of the halfplane that does not include P_i, P_j. The hull edges sorted by the circular ordering of all such normal vectors define a closed polygonal chain called the *cluster hull* of $\mathcal{P}$, denoted $CLH(\mathcal{P})$.

We show that there is a one-to-one correspondence between the unbounded faces of the FCVD and the HVD. Therefore, results for hulls [16] directly follow.

▶ **Lemma 2.2.** *A region $f_c reg(p)$ is unbounded if and only if p is a vertex of $CLH(\mathcal{P})$. The circular order of hull edges along $CLH(\mathcal{P})$ is equal to that of unbounded edges of* $\mathrm{FCVD}(\mathcal{P})$.

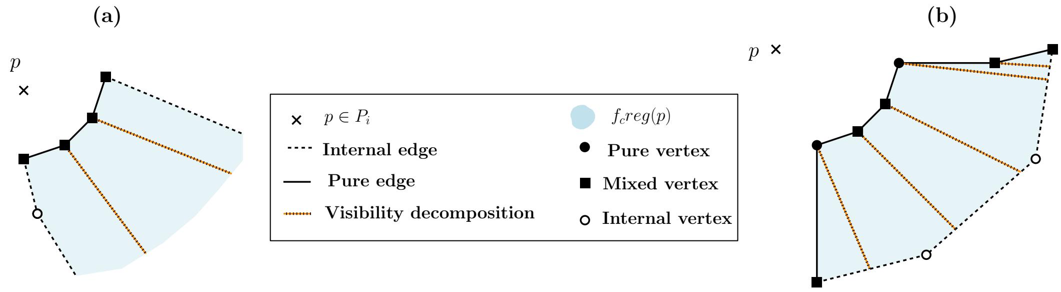

Figure 3 (a) An unbounded and (b) a bounded face of a point $p \in P_i$.

3 Conditions for linear-size diagrams

Abstract Voronoi diagrams were introduced by Klein [12]. Instead of sites and distance measures, these diagrams are defined in terms of bisecting curves satisfying a set of simple combinatorial properties, called axioms. In the context of color Voronoi diagram, these axioms can be stated as follows, for every subset $\mathcal{P}' \subseteq \mathcal{P}$:

(A1) Each region in NCVD($\mathcal{P}'$) is non-empty and path-wise connected.
(A2) Each point in the plane belongs to the closure of a region in NCVD($\mathcal{P}'$).
(A3) Each color bisector is an unbounded Jordan curve.
(A4) Any two color bisectors intersect transversally and in a finite number of points.

A family of clusters is called *admissible* if the system of bisectors satisfies (A1)-(A4). By the structural properties of farthest abstract Voronoi diagrams [5, 14] we derive the following.

▶ **Lemma 3.1.** *If $\mathcal{P}$ is admissible, then the skeleton of* FCVD($\mathcal{P}$) *is a tree of $O(n)$ total structural complexity. One region may consist of $\Theta(m)$ disjoint faces and the total number of faces is $O(m)$.*

Two clusters are called *linearly-separable* if they have disjoint convex hulls. A family of pairwise linearly-separable clusters is also called *linearly-separable*. The color bisector of two linearly-separable clusters is a single unbounded, monotone chain. The color bisectors of three pairwise linearly-separable clusters, however, $b_c(P_i, P_j)$ and $b_c(P_j, P_k)$ may intersect $\Theta(|P_i| + |P_j| + |P_k|)$ times (see Fig. 2b). Thus, a linearly separable family need not be admissible. By showing that if the regions of NCVD($\mathcal{P}$) are connected then the same should hold for NCVD($\mathcal{P}'$), for any $\mathcal{P}' \subseteq \mathcal{P}$, we derive the following.

▶ **Lemma 3.2.** *Let $\mathcal{P}$ be a linearly-separable family of clusters. If the regions in* NCVD($\mathcal{P}$) *are path-connected, then $\mathcal{P}$ is admissible.*

Lemma 3.2 indicates that we can determine if a family $\mathcal{P}$ is admissible in $O(n \log n)$ time. A family of clusters $\mathcal{P}$ is called *disk-separable* if for every cluster $P_i \in \mathcal{P}$ there exists a disk containing P_i and no point from other clusters (see Fig. 4). By proving that disk separability implies connected regions in NCVD($\mathcal{P}$), we derive:

▶ **Lemma 3.3.** *Any family of disk-separable clusters $\mathcal{P}$ is admissible.*

We now look into linearly-separable families of clusters. The following statement has been proven for clusters of cardinality two [6] but holds also for general clusters.

▶ **Lemma 3.4.** *If $\mathcal{P}$ is linearly-separable, then* FCVD($\mathcal{P}$) *has $O(m)$ unbounded faces.*

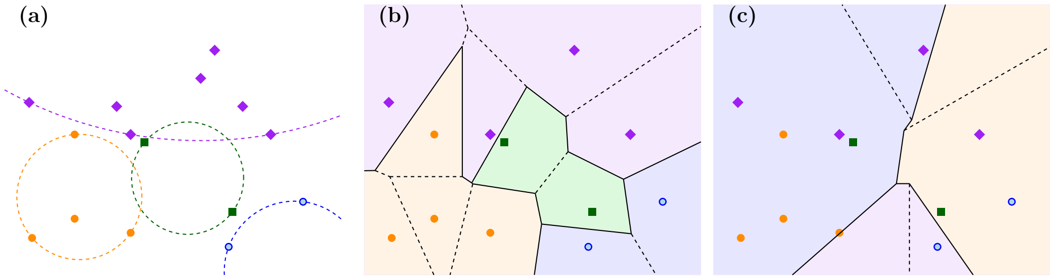

Figure 4 (a) A disk-separable family of clusters $\mathcal{P}$ along with (b) NCVD($\mathcal{P}$) and (c) FCVD($\mathcal{P}$).

A pair of points $(p_1, p_2) \in P_i$, which defines a Voronoi edge e in Vor(P_i), is said to be *straddled* by a cluster $Q_j \in \mathcal{P}$ if the line through (p_1, p_2) intersects the segment $\overline{q_1 q_2}$ defined by $(q_1, q_2) \in Q_j$ and the circles through (q_1, p_1, p_2) and (q_2, p_1, p_2) are both centered on e (see Fig. 5a). We also say that (q_1, q_2) and Q_j straddle the Voronoi edge e.

We define the *straddling number* of e, denoted $s(e)$, as the number of clusters in $\mathcal{P}$ that straddle e. Clearly, for a cluster P_i, $s(P_i) = O(m|P_i|)$. The *straddling number of family* $\mathcal{P}$, is $s(\mathcal{P}) = \sum_{P_i \in \mathcal{P}} s(P_i)$. In the worst case, $s(\mathcal{P}) = \Theta(mn)$.

▶ **Lemma 3.5.** *If $\mathcal{P}$ is linearly-separable, then the number of bounded faces, and the overall structural complexity of* FCVD($\mathcal{P}$)*, is $O(n + s(\mathcal{P}))$.*

Proof. *(sketch)* For each Voronoi edge e of Vor(P_i) we allow one bounded face of $f_c reg(P_i)$ and count the number of mixed vertices that may be incident to additional faces of $f_c reg(P_i)$ on e. Let v_1, v_2 be two consecutive mixed vertices on a Voronoi edge e of Vor(P_i), induced by points (p_1, p_2), such that segment $\overline{v_1 v_2} \notin f_c reg(P_i)$ (see Fig.5). Suppose v_1 is induced by $q_1 \in Q_j$. By considering a disk moving from left to right on e and touching (p_1, p_2), we can show that v_2 must be induced by a point $q_2 \in Q_j$ such that (q_1, q_2) defines a straddle on e. In addition, cluster Q_j cannot induce any other mixed vertex on e. Thus, the pair of vertices (v_1, v_2) is charged to a unique cluster counted in the straddling number of e. ◀

By Lemma 3.5, if the straddling number $s(\mathcal{P})$ is $O(n)$, then FCVD($\mathcal{P}$) has complexity $O(n)$.

4 Construction algorithms

Consider a divide & conquer approach. Split $\mathcal{P}$ into $\mathcal{P}_L$ and $\mathcal{P}_R$ by a vertical line; Compute FCVD($\mathcal{P}_L$) and FCVD($\mathcal{P}_R$) recursively; Merge FCVD($\mathcal{P}_L$) and FCVD($\mathcal{P}_R$) to obtain FCVD($\mathcal{P}$). Merging requires constructing the *merge curve* $\mathcal{M}(\mathcal{P}_L \cup \mathcal{P}_R)$, which is the set of pure edges of FCVD($\mathcal{P}_L \cup \mathcal{P}_R$) belonging to bisectors $b_c(P_i, P_j)$ with $P_i \in \mathcal{P}_L$ and $P_j \in \mathcal{P}_R$. A merge curve may consist of linearly many chains, called *components*. To construct it, a *starting point* has to be found on each component and then the chain has to be *traced*.

Given a starting point on a component we can efficiently trace it, by adapting standard tracing methods and exploiting the visibility decomposition, similarly to [16].

▶ **Lemma 4.1.** *Given diagrams* FCVD($\mathcal{P}_L$), FCVD($\mathcal{P}_R$) *and a starting point on a component M of $\mathcal{M}(\mathcal{P}_A, \mathcal{P}_B)$, the component M can be computed in $O(|M|)$ time.*

Due to Lemma 2.2, we can identify starting points on the unbounded components of $\mathcal{M}(\mathcal{P}_A, \mathcal{P}_B)$ by merging $CLH(\mathcal{P}_L)$ and $CLH(\mathcal{P}_R)$, before merging the two diagrams. This can be done in time $O(|CLH(\mathcal{P}_L)| + |CLH(\mathcal{P}_R)|)$, see [16].

If $\mathcal{P}$ is admissible, (such as a family of disk separable clusters), then all regions of

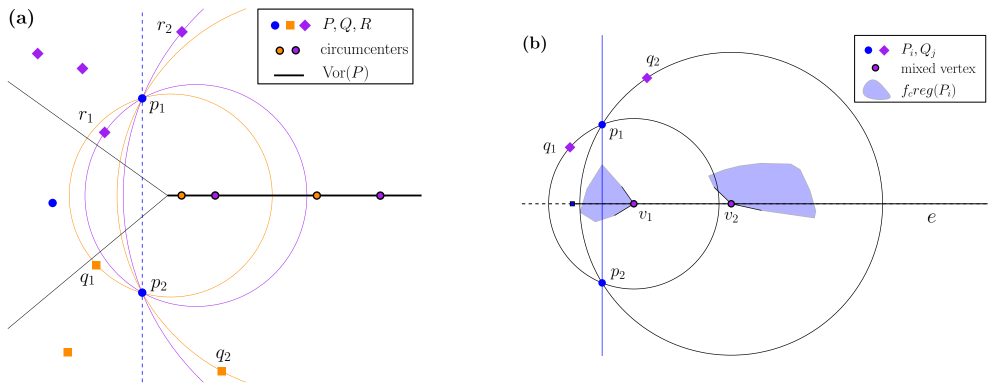

Figure 5 (a) A family $\mathcal{P}$, where pair (p_1, p_2) is straddled by two clusters Q, R.(b) Illustration of the proof of Lemma 3.5.

FCVD($\mathcal{P}$) are unbounded (Lemma 3.1) and this is true for all components of the merge curve. Thus, we derive the following.

▶ **Theorem 1.** If $\mathcal{P}$ is admissible, then FCVD($\mathcal{P}$) can be constructed in $O(n \log n)$ time.

Note that for an admissible family $\mathcal{P}$, FCVD($\mathcal{P}$) could also be computed using the randomized algorithm of [14] for abstract Voronoi diagrams. Color bisectors, however, may have $\Theta(n)$ complexity, so, a direct application would give time complexity $O(n^2 \log n)$.

When $\mathcal{P}$ is not admissible, the challenge is to identify starting points on the bounded components of the merge curve. For linearly-separable families where clusters have a constant straddling number, there are constant number of bounded components. To identify starting points on these components, the data structure and technique of [10] can be used to do this in $O(n \log n)$ time, yielding an $O(n \log^2 n)$-time algorithm.

▶ **Theorem 2.** If $\mathcal{P}$ is a linearly-separable family of clusters, where $s(P_i)$ is constant for any $P_i \in \mathcal{P}$, then FCVD($\mathcal{P}$) can be constructed in $O(n \log^2 n)$ time.

We conjecture that for linearly-separable families the FCVD can have complexity $\Theta(mn)$.

References

1 Manuel Abellanas, Ferran Hurtado, Christian Icking, Rolf Klein, Elmar Langetepe, Lihong Ma, Belén Palop, and Vera Sacristán. The farthest color Voronoi diagram and related problems. Technical report, Rheinische Friedrich–Wilhelms–Universität Bonn, 2006.
2 Franz Aurenhammer, Rolf Klein, and Der-Tsai Lee. *Voronoi Diagrams and Delaunay Triangulations*. World Scientific, 2013.
3 Sang Won Bae. On linear-sized farthest-color Voronoi diagrams. *IEICE Transactions on Information and Systems*, 95(3):731–736, 2012.
4 Sang-Won Bae, Chun-Seok Lee, and Sung-Hee Choi. On exact solutions to the euclidean bottleneck steiner tree problem. *Information Processing Letters*, 110(16):672–678, 2010.
5 Cecilia Bohler, Panagiotis Cheilaris, Rolf Klein, Chih-Hung Liu, Evanthia Papadopoulou, and Maksym Zavershynskyi. On the complexity of higher order abstract Voronoi diagrams. *Computational Geometry*, 48(8):539 – 551, 2015.
6 Mercè Claverol, Elena Khramtcova, Evanthia Papadopoulou, Maria Saumell, and Carlos Seara. Stabbing circles for sets of segments in the plane. *Algorithmica*, pages 1–36, 2017.

7 Frank Dehne, Anil Maheshwari, and Ryan Taylor. A coarse grained parallel algorithm for hausdorff voronoi diagrams. In *Parallel Processing, 2006. ICPP 2006. International Conference on*, pages 497–504. IEEE, 2006.

8 Herbert Edelsbrunner, Leonidas J Guibas, and Micha Sharir. The upper envelope of piecewise linear functions: algorithms and applications. *Discrete & Computational Geometry*, 4(1):311–336, 1989.

9 Daniel P Huttenlocher, Klara Kedem, and Micha Sharir. The upper envelope of Voronoi surfaces and its applications. *Discrete & Computational Geometry*, 9(3):267–291, 1993.

10 John Iacono, Elena Khramtcova, and Stefan Langerman. Searching edges in the overlap of two plane graphs. *arXiv preprint arXiv:1701.02229*, 2017.

11 Elena Khramtcova and Evanthia Papadopoulou. Randomized incremental construction for the Hausdorff Voronoi diagram revisited and extended. In *International Computing and Combinatorics Conference*, pages 321–332. Springer, 2017.

12 Rolf Klein. *Concrete and abstract Voronoi diagrams*, volume 400. Springer Science & Business Media, 1989.

13 Chunseok Lee, Donghoon Shin, Sang Won Bae, and Sunghee Choi. Best and worst-case coverage problems for arbitrary paths in wireless sensor networks. *Ad Hoc Networks*, 11(6):1699–1714, 2013.

14 Kurt Mehlhorn, Stefan Meiser, and Ronald Rasch. Furthest site abstract Voronoi diagrams. *International Journal of Computational Geometry & Applications*, 11(06):583–616, 2001.

15 Evanthia Papadopoulou. The Hausdorff Voronoi diagram of point clusters in the plane. *Algorithmica*, 40(2):63–82, 2004.

16 Evanthia Papadopoulou and Der-Tsai Lee. The Hausdorff Voronoi diagram of polygonal objects: A divide and conquer approach. *International Journal of Computational Geometry & Applications*, 14(06):421–452, 2004.