Triangles in the colored Euclidean plane*

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Abstract

We study a variation of the well known Hadwiger-Nelson problem on the chromatic number of the Euclidean plane. An embedding of a given triangle T into the colored plane is called monochromatic, if the three corners of the triangle get the same color. We provide a classification of triangles according to the number of colors needed to color the plane so that the triangle can not be embedded monochromatically. For example, we show that for near-equilateral triangles three colors are enough and that for almost all triangles six colors are sufficient.

1 Introduction

An *r-coloring* of the Euclidean plane is a partition such that every point of the plane gets one of r colors assigned. The famous Hadwiger-Nelson problem (see Soifer [10] for an extensive history) asks for the minimum r to r-color the Euclidean plane so that every two points at unit distance from each other have different colors. It was well known that this so-called chromatic number of the plane is at least 4, and that 7 colors are for sure sufficient. Most recently de Grey [1] constructed an explicit unit-distance graph with 1581 vertices with chromatic number 5. This was further optimized by Heule [5] to graphs with 'only' 553 vertices. So the chromatic number of the Euclidean plane is now 5, 6 or 7.

To shed more light on the Hadwiger-Nelson problem, Graham [3] posed the following question.

▶ Problem 1. What is the smallest number c, so that for every triangle T, there exists a c-coloring of the Euclidean plane so that it is not possible to embed T in such a way that all three vertices of T have the same color?

If a triangle T, or its reflected copy, can be embedded in the colored plane such that all three vertices have the same color, we call the embedding and the triangle monochromatic, and non-monochromatic otherwise. If the color of all three vertices is for example red, we also call the triangle red.

As for a lower bound on c it has been shown that some triangles exist monochromatically in every 2-coloring of the plane [2, 9]. For example in [2] they show that a triangle with smallest side of length 1 and angles 30°, 60°, and 90° is monochromatic in every 2-coloring. The following argumentation for this statement is based on Soifer [10]. As in any 2-coloring an equilateral triangle has two vertices with the same color, there always exist two points with the same color, say blue, with distance 2 (or any other fixed distances). Consider the corners of a regular hexagon where these two blue points are antipodal, see Figure 1. If any other corner of the hexagon is also blue, we are done. Otherwise, the remaining 4 corners are all red, and we have the required triangle in red. Surprisingly, this example constitutes the best known lower bound, and thus Graham conjectured that $c = 3$ could be the true bound.

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This is an extended abstract of a presentation given at EuroCG'19. It has been made public for the benefit of the community and should be considered a preprint rather than a formally reviewed paper. Thus, this work is expected to appear eventually in more final form at a conference with formal proceedings and/or in a journal.

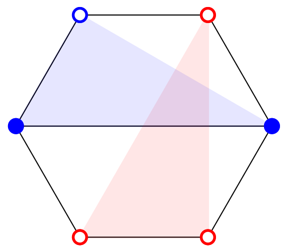


Figure 1 A triangle with 30°, 60°, and 90° always exists monochromatically in every 2-coloring.

Obviously an upper bound for the chromatic number of the plane is also an upper bound for c, and thus we have $c \in \{3, 4, 5, 6, 7\}$.

In this work we concentrate our considerations on specific triangles and how many colors guarantee their non-monochromatic embedding.

▶ Problem 2. Given a triangle T. What is the smallest number $c(T)$, so that we can $c(T)$-color the Euclidean plane, such that T can not be embedded monochromatically.

An upper bound on c in Problem 1 is also an upper bound for all $c(T)$ in Problem 2, and if we can show an upper bound for $c(T)$ for all triangles T then this implies an upper bound for c. For example in [6] it has been shown that the equilateral triangle with side length s is non-monochromatic in a 2-coloring with halfopen horizontal strips of height $\frac{\sqrt{3}}{2}s$. We call such a coloring *zebra* coloring. Also other colorings for equilateral triangles are provided in [6]. Pritikin [8] used colorings, where points with the same color and distance 1 only occur in one color. Modifying his 7-coloring so that it becomes a proper 6-coloring of the plane leads to results which are slightly worse than what we will go to present. In addition, in his 5-coloring almost all triangles occure monochromatic. Graham and Tressler [4] show bounds for degenerated triangles and mentioned that zebra colorings avoid a large class of triangles, without giving explicit bounds.

1.1 Results

In Section 2 we will consider triangles for which a 6-coloring exists such that a monochromatic embedding is not possible. We show that this holds for almost all triangles, thus lowering the upper bound for Problem 1 from 7 to 6 with the exception of almost isosceles triangles with a short base. In Section 3 we categorize triangles according to the number of colors which are needed to guarantee that a monochromatic embedding is not possible. We show that for near equilateral triangles 3 colors are sufficient. The flatter a triangle gets, the more colors are needed. Due to space limitations several proofs will be omitted in this note.

2 Non-monochromatic triangles in the 6-colored plane

Let T be a triangle with vertices A, B and C. We write AB for the line segment AB and $\overline{AB}$ for its length. Without loss of generality we assume that AB is the longest side of T.

▶ **Definition 2.1.** We denote the heights of T as depicted in Figure 2. We call T a normed triangle, if the longest side AB of T has length 1, and $\overline{BC} \leq \overline{AC}$ holds.

Observe that a triangle T with side lengths a, b and c is non-monochromatic in some r-coloring, if and only if there exists an r-coloring F, so that the triangle T_a with side lengths 1, $\frac{b}{a}$ and $\frac{c}{a}$ is non-monochromatic in F. Therefore we can restrict our investigations

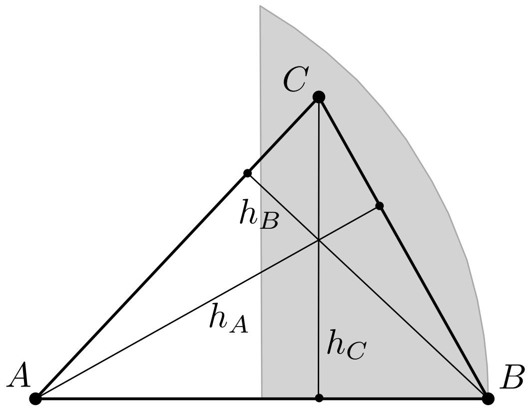


Figure 2 Normed triangle with $\overline{BC} \leq \overline{AC} \leq \overline{AB} = 1$. The grey area shows all possible locations for vertex C.

to normed triangles. The grey area in Figure 2 shows all possible locations of vertex C to be considered.

2.1 Zebra colorings

In a *zebra coloring* the plane is cyclically colored with horizontal strips, all of the same height. The strips are halfopen, that is, the boundary between two neighboured strips has the same color as the strip above it. For a normed triangle T we have that h_C is the shortest height of T. Thus the height of the strips has to be at most h_C, as otherwise T fits into one strip and obviously would be monochromatic. So assume that the height of each strip is exactly h_C. This implies that the vertices of the triangle have to lie in at least two different strips.

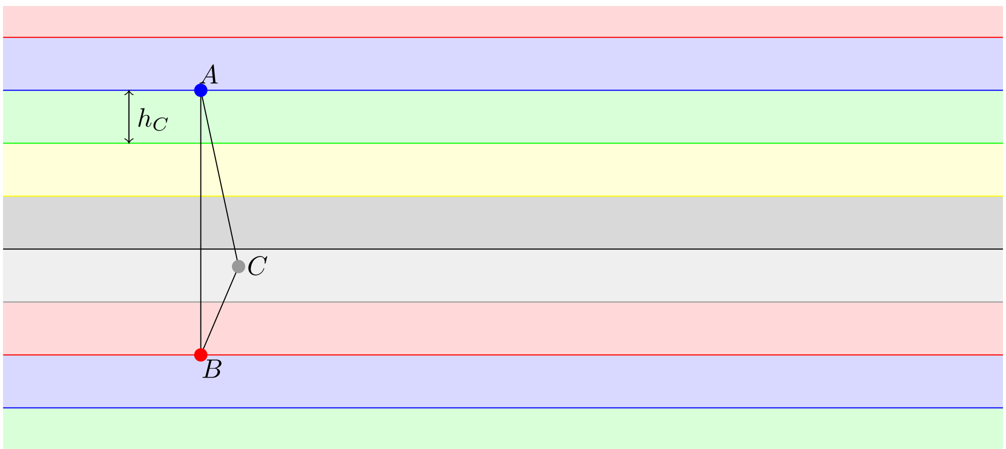


Figure 3 Zebra coloring with 6 colors with halfopen strips of height h_C

For 6 colors the distance between two points in two different strips with the same color is strictly larger than $5h_C$, because there are five strips between two strips with the same color. So if two points have the same color and have distance at most $5h_C$, then these two points have to lie in the same strip. Similarly, if a point of the triangle is the only one in a strip, then the distances to both other points are at least $5h_C$ if all the points are of the same color. Therefore a normed triangle T is non-monochromatic if the second longest side $\overline{AC} \leq 5h_C$ and we obtain the following lemma.

▶ **Lemma 2.2.** *All normed triangles with $\overline{AC} \leq 5h_C$ are non-monochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

For triangles where every angle is at most 90° we can slightly improve this result. The basic idea is to place the triangle between two strips of the same color such that the longest height h_A is vertical. If we rotate the triangle around the vertex A until one of the sides AB or AC is vertical, then one of the two vertices B and C moves upwards while the other one moves downwards, and eventually one of them has to leave the strip. The details of the proof are omitted.

▶ **Theorem 2.3.** *Every normed triangle, in which every angle is at most $90°$ and $h_A \leq 5h_C$ is non-monochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

By comparing different expressions of the area of the triangle, we get the following corollary.

▶ **Corollary 2.4.** *Every normed triangle, in which every angle is at most $90°$, and $\overline{BC} \geq \frac{1}{5}$ is non-monochromatic in a zebra coloring with 6 colors, where all strips have height h_C.*

2.2 Hexagon colorings

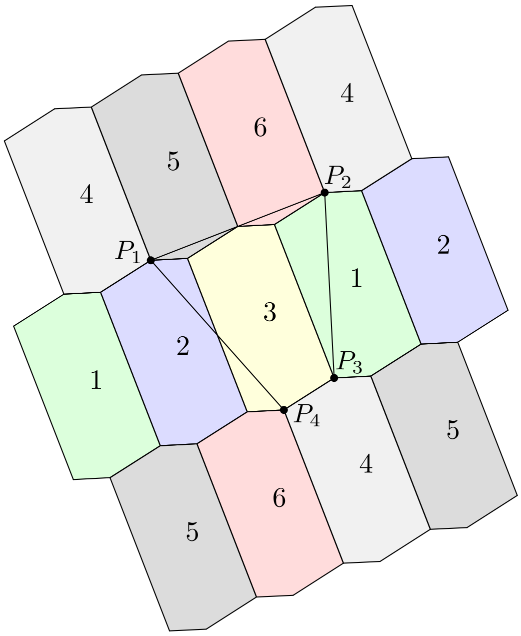


Figure 4 6-coloring with hexagons, where all diagonals have length 1 and opposing sides are parallel

Consider the hexagonal 6-coloring shown in Figure 4. The vertices of a hexagon lie on a circle with radius $\frac{1}{2}$ and are center symmetric. Thus the three central diagonals have length 1 and opposing sides are parallel. The hexagons are halfopen in the sense that we color the two lowest vertices of a hexagon as well as the three sides, which are incident to these

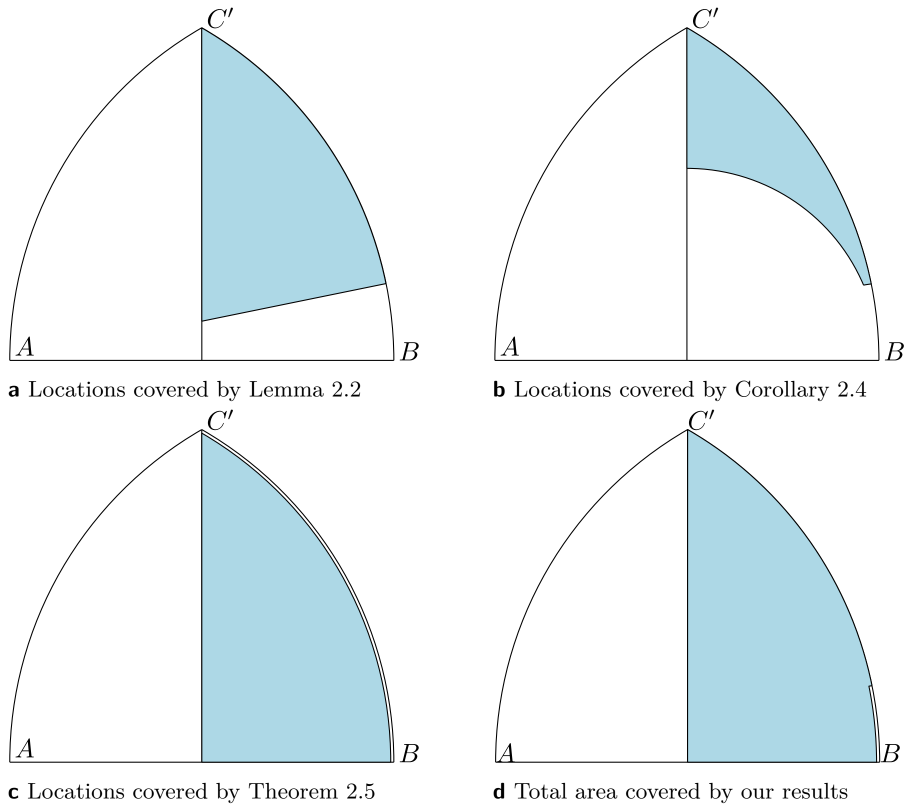


Figure 5 Possible locations for vertex C so that there exists a 6-coloring, such that the triangle ABC is non-monochromatic.

vertices, with the same color as the hexagon. For example in Figure 4 the vertex P_4 and side P_3P_4 have color 3, but P_3 has color 1.

Our goal is to maximize the shortest of the lengths $\overline{P_1P_2}$, $\overline{P_2P_3}$ and $\overline{P_1P_4}$, as they are also the lower bounds for the distance of two points in different hexagons with the same color. Details on how to compute the exact lengths of the hexagons will be given in the full version of this work.

▶ **Theorem 2.5.** *All normed triangles T with $\overline{AC} \leq 0.992076$ are non-monochromatic in the hexagon coloring in Figure 4 with specific lengths.*

2.3 Summarizing bounds

Let us look which types of normed triangles are covered by the previous results. Recall that the grey area in Figure 2 shows the relevant region of vertex C. Figures 5a to 5c show which locations are covered by our results, and Figure 5d gives their union.

For almost all triangles T there exists a 6-coloring so that T is non-monochromatic. Only for near-isosceles triangles with $0.992076\overline{AB} < \overline{AC} \leq \overline{AB}$ and $\overline{BC} < \frac{1}{5}\overline{AB}$ we have no such 6-coloring. This has to be seen in context of the Hadwiger-Nelson problem. For these triangles the vertex C is rather close to the end of the unit length segment AB, meaning that in the extreme case these triangles approach the segment.

Furthermore, Figure 5c shows that Theorem 2.5 covers almost all possible choices of C. For locations of C which are not covered by Theorem 2.5 we see that Corollary 2.4 covers slightly more than Lemma 2.2. Actually Lemma 2.2 is not needed for 6-colorings, as for almost isosceles triangles, Corollary 2.4 is better than Lemma 2.2.

3 Non-monochromatic triangles with fewer colors

We have shown that for most triangles there exists a 6-coloring that prevents a monochromatic embedding. A natural question is to ask for which triangles less colors are sufficient. We can use zebra colorings to generalize Lemma 2.2 and Corollary 2.4. Similar to Lemma 2.2 we get.

▶ **Theorem 3.1.** *All normed triangles with $\overline{AC} \le (k-1)h_C$ are non-monochromatic in a zebra coloring with k colors, $3 \le k \le 6$, where all strips have height h_C.*

Corollary 2.4 generalizes to

▶ **Theorem 3.2.** *All normed triangles for which every angle is at most 90° and $\overline{BC} \ge \frac{1}{k-1}$ are non-monochromatic in a zebra coloring with k colors, $2 \le k \le 6$. where all strips have height h_C.*

Since our arguments for the proofs of Lemma 2.2, Theorem 2.3 and Corollary 2.4 were about the distance between two points with the same color in different strips, we can prove these statements in a similar way. Actually we just need to replace 5 by $k-1$. Note that Theorem 3.2 also works for 2 colors, whereas Theorem 3.1 only works for 3 or more colors.

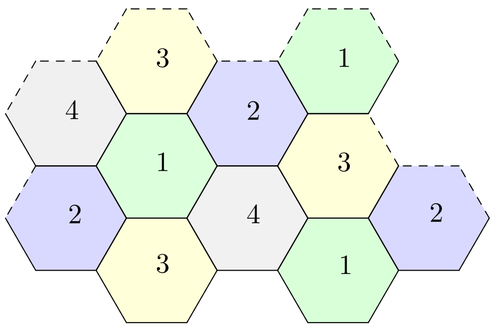
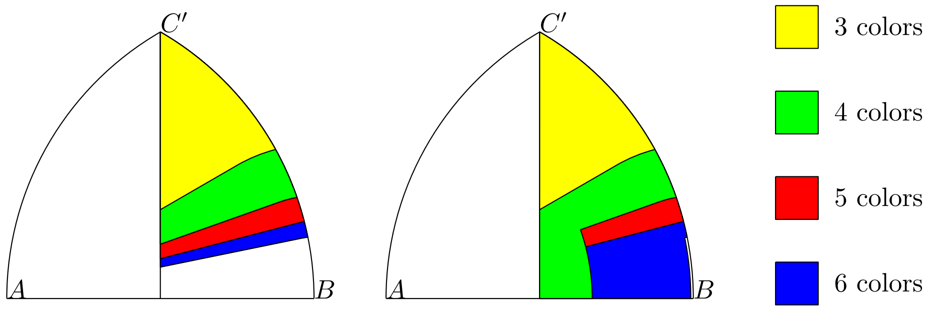


Figure 6 4-coloring with regular hexagons of diameter 1

In the 4-coloring in Figure 6 all hexagons have diameter 1. The three sides and two vertices below have the same color as the hexagon. This gives us the following theorem (proof omitted).

▶ **Theorem 3.3.** *In the 4-coloring in Figure 6, all normed triangles with $\overline{AC} \le \frac{\sqrt{3}}{2}$ are non-monochromatic.*

As before we summarize the results in a diagam, see Figure 7. For all possible locations of the third vertex C of a normed triangle the colors indicate which k is sufficient so that a k-coloring exists such that the triangle is non-monochromatic in this coloring. For example, if C is in the yellow shaded area, then there exists a 3-coloring such that the triangle ABC is non-monochromatic. This includes all triangles where the length of the three edges does not differ too much. For Figure 7a only zebra colorings are used, while for Figure 7b also hexagonal colorings are allowed.



a Only zebra colorings **b** All colorings **c** Color code

Figure 7 Location of vertex C of triangles for which a k-coloring, $3 \le k \le 6$ exists so that they are non-monochromatic.

4 Conclusion

We have shown that for almost all triangles there exists a 6-coloring, such that the triangle is non-monochromatic in this coloring. To be precise, for every normed triangle with $\overline{AC} \le 0.992076$ or $\overline{BC} \ge \frac{1}{5}$ we can give such a 6-coloring. We have also seen that some triangles are non-monochromatic in colorings with less than 6 colors. It remains an open problem, if for every triangle there exists a 6-coloring of the plane, such that the triangle is non-monochromatic. Or more generally, what is the smallest c so that there exists for every triangle a c-coloring, such that the triangle is non-monochromatic.

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